Chapter VI

MALTHUSIAN POPULATIONS WITH KNOWN AGE DISTRIBUTION OF DEATHS: THE SUB-SETS G(r) AND PROCESSES OF DEMOGRAPHIC EVOLUTION WITH CONSTANT AGE DISTRIBUTION OF DEATHS

In this chapter, we propose to study the sub-sets G(r) described above. We assume that the age distribution of deaths, $d_0(a)$ is constant and given, while the mortality function and the age distribution of the population are constant but are not given. All Malthusian populations corresponding to these conditions constitute the sub-sets $G_0(r)$ linked to the age distribution of deaths $d_0(a)$. By varying $d_0(a)$ we obtain the series of sub-sets G(r).

The basic relations used for studying the sub-sets G(r) are the same as those used for the study of the preceding sub-sets H(r) and F(r). We shall present them once again below, in connexion with the determination in a given set $G_0(r)$ of a population satisfying a given condition.

A. Population of a sub-set $G_0(r)$ satisfying certain conditions

First example: the rate of natural variation r is given

The survivorship function is written:

$$p(a) = 1 - \frac{\int_0^a d(a)e^{ra}da}{\int_0^\omega d(a)e^{ra}da} \qquad \text{(II.10)}$$

(This is formula II.10 in chapter II.)

For any given value of r the quantity:

$$\frac{\int_0^a d(a)e^{ra}da}{\int_0^\omega d(a)e^{ra}da}$$

increases from 0 to 1 when a increases from 0 to ω.

Consequently, formula (II.10) is valid for all values of r.

Table VI.1 gives details of the calculation for applying this formula in the following case: the function $d(a)$ is the age distribution of deaths calculated in chapter II (table II.3). It corresponds to a Malthusian population with a rate of natural variation of $r = 0.03$, whose mortality is that of the intermediate model life table with an expectation of life at birth for both sexes of fifty years. Here we disregard the way in which this age distribution of deaths was calculated, taking it as an established fact, and we proceed to consider the sub-set $G_0(r)$ linked to this age distribution.

We propose to determine the population of this set $G_0(r)$, which has a rate of natural increase of $r = 0.03$. By applying formula (II.10), we should again find the function $p(a)$ which was used in the calculation of $d(a)$, and this is in fact what table VI.1 shows.

Once we know the survivorship function $p(a)$, it is easy to calculate all the other characteristics of the population.

The fact that formula II.10 is written in continuous notation and that we have the age distribution of deaths only in discontinuous terms causes no difficulty here, as it did in the case of the study of the sub-sets F(r). It was simply assumed, for the purpose of calculating the integrals of formula II-10, that for the 20-24 age group, for example, we had:

$$\int_{20}^{25} d(a)e^{ra}da \# e^{22.5r}d_{20\text{-}24}$$

Tables VI.2 and VI.3 give computations similar to those in table VI.1 for the other two values of the rate of natural increase: $r = 0.015$ and $r = 0$.

Table VI.5 gives the age distributions of the populations.

We shall see in a moment how these computations can be used for the three different rates of variation.

Second example: the crude birth rate b_0 is given

As was seen above, for each value of r there is a survivorship function $p(a)$ and, consequently, a crude birth rate:

$$b_0 = \frac{1}{\int_0^\omega e^{-ra}p(a)da}$$

We can therefore regard the crude birth rate b as a function of r. The three computations in tables VI.1, VI.2 and VI.3 give the values of this function for $r = 0.03$, $r = 0.015$ and $r = 0$. If we prepare a graph with r on the horizontal axis and b on the vertical axis, we have three points on the curve representing $b(r)$. Graph VI.1 was drawn in this way. We see that, when r varies from $-\infty$ to $+\infty$, b passes through a minimum b_m.

If $b_0 > b_m$, the straight line of the ordinate b_0 intersects the curve G(r) at two points M_1 and M_2, having as their abscissae r_1 and r_2. There are therefore two Malthusian populations corresponding to the rates of natural variation r_1 and r_2. Thus, for the purpose of calculating the characteristics of these populations, we are brought back to the first example, since we know r_1 and r_2.

If $b_0 < b_m$, there is no solution to the problem, and if b_0 equals b_m there is one double solution.

TABLE VI.1. COMPUTATION OF THE FEMALE SURVIVORSHIP FUNCTION WHICH, WHEN ASSOCIATED WITH A RATE OF NATURAL VARIATION OF 0.03, LEADS TO A FEMALE MALTHUSIAN POPULATION HAVING A DISTRIBUTION OF DEATHS BY AGE GROUPS IDENTICAL WITH THE DISTRIBUTION d_a SHOWN IN THE THIRD COLUMN OF THE TABLE

Median age a	*Age group (years)* a	*Distribution of deaths by age groups* d_a	e^{-ra} *for* $r = 0.03$	*Quotient of the two preceding columns* $e^{ra}d_a$	*Cumulative totals of age groups* $\int_0^a d(a)e^{ra}da$	*Distribution by age groups of the cumulative totals*	*Survivors at the beginning of each age group*	*Survivors in each age group*	*Initial survivorship function*
0.5	Under 1	366 651	0.98511	372 193	372 193	12 372	100 000	90 721	100 000
3.0	1-4	150 864	0.91393	165 072	537 265	17 859	87 628	339 008	87 625
7.5	5-9	38 967	0.79852	48 799	586 064	19 481	82 150	406 672	82 136
12.5	10-14	24 421	0.68729	35 532	621 596	20 663	80 519	399 640	80 515
17.5	15-19	30 560	0.59156	51 660	673 256	22 380	79 337	392 392	79 333
22.5	20-24	34 964	0.50916	68 670	741 926	24 663	77 620	382 392	77 615
27.5	25-29	31 494	0.43824	71 865	813 791	27 051	75 337	370 715	75 332
32.5	30-34	27 691	0.37719	73 414	887 205	29 492	72 949	358 642	72 940
37.5	35-39	24 621	0.32465	75 839	963 044	32 013	70 508	346 237	70 500
42.5	40-44	22 820	0.27943	81 666	1 044 710	34 727	67 987	333 150	67 981
47.5	45-49	23 153	0.24051	96 266	1 140 976	37 927	65 273	318 365	65 266
52.5	50-54	24 755	0.20701	119 584	1 260 560	41 903	62 073	300 425	62 064
57.5	55-59	26 890	0.17817	150 923	1 411 483	46 919	58 097	277 942	58 093
62.5	60-64	30 760	0.15336	200 574	1 612 057	53 587	53 080	248 732	53 076
67.5	65-69	34 430	0.13199	260 853	1 872 910	62 258	46 413	210 387	46 413
72.5 . . .	70-74	36 231	0.11361	318 907	2 191 817	72 859	37 742	162 207	37 747
77.5	75-79	32 495	0.9778	332 328	2 524 145	83 906	27 141	108 087	27 141
82.5	80-84	22 685	0.8416	269 546	2 793 691	72 866	16 094	58 070	16 086
87.5	85 +	15 547	0.7244	214 619	3 008 310	100 000	7 134	27 480	7 123

TABLE VI.2. COMPUTATION OF THE FEMALE MORTALITY FUNCTIONS WHICH, WHEN ASSOCIATED WITH A RATE OF NATURAL VARIATION OF 0.015, LEADS TO A FEMALE MALTHUSIAN POPULATION HAVING A DISTRIBUTION OF DEATHS BY AGE GROUPS IDENTICAL WITH THE DISTRIBUTION d_a SHOWN IN THE THIRD COLUMN OF THE TABLE

Median age a	*Age group (years)* a	*Distribution of deaths by age groups* d_a	$e^{-ra}da$ *for* $r = 0.015$	*Quotient of the two preceding columns* $e^{-ra}d_a$	*Cumulative totals of age groups*	*Distribution by age groups of the cumulative totals*	*Survivors at the beginning of each age group*	*Deaths from one age to the next*	*Survivors in each age group* L_a	*Death rate (per 1 000)* m_a
0.5 . .	Under 1	366 651	1.00750	369 401	369 401	23 769	100 000	23 769	82 713	287.37
3.0 . .	1-4	150 864	1.04600	157 804	527 205	33 922	76 231	10 153	283 603	35.80
7.5 . .	5-9	38 967	1.11917	43 611	570 816	36 728	66 078	2 806	323 375	8.68
12.5 . .	10-14	24 421	1.20623	29 457	600 273	38 624	63 272	1 896	311 620	6.08
17.5 . .	15-19	30 560	1.30128	39 767	640 040	41 182	61 376	2 558	300 485	8.51
22.5 . .	20-24	34 964	1.40144	49 000	689 040	44 335	58 818	3 153	286 208	11.02
27.5 . .	25-29	31 494	1.51059	47 575	736 615	47 396	55 665	3 153	270 673	11.31
32.5 . .	30-34	27 691	1.62829	45 089	781 704	50 298	52 604	2 902	255 765	11.35
37.5 . .	35-39	24 621	1.75515	43 214	824 918	53 078	49 702	2 780	241 560	11.51
42.5 . .	40-44	22 820	1.89174	43 170	868 088	55 856	46 922	2 778	227 665	12.20
47.5 . .	45-49	23 153	2.03918	47 213	915 301	58 894	44 144	3 038	213 125	14.25
52.5 . .	50-54	24 755	2.19899	54 436	969 737	62 396	41 106	3 502	196 775	17.80
57.5 . .	55-59	26 890	2.36918	63 707	1 033 444	66 495	37 604	4 099	177 773	23.06
62.5 . .	60-64	30 760	2.55369	78 552	1 111 996	71 550	33 505	5 055	154 888	32.64
67.5 . .	65-69	34 450	2.75257	94 771	1 206 767	77 648	28 450	6 098	127 005	48.01
72.5 . .	70-74	36 231	2.96685	107 492	1 314 259	84 564	22 352	6 916	94 470	73.21
77.5 . .	75-79	32 495	3.19792	103 916	1 418 175	91 250	15 436	6 686	60 465	110.58
82.5 . .	80-84	22 686	3.44709	78 201	1 496 376	96 282	8 750	5 032	31 170	161.44
87.5 . .	85 +	15 547	3.71655	57 781	1 554 157	100 000	3 718	3 718	13 274	280.10

TABLE VI.3. COMPUTATION OF FEMALE MORTALITY FUNCTIONS SUCH THAT IN THE CORRESPONDING STATIONARY POPULATION DISTRIBUTION OF DEATHS BY AGE GROUPS IS IDENTICAL WITH THE DISTRIBUTION d_a SHOWN IN THE THIRD COLUMN OF THE TABLE

Median age α	*Age group (years)* a	*Distribution of deaths by age groups* d_a	*Cumulative totals*	*Survivors at the beginning of each age group*	*Deaths from one age to the next*	*Survivors in each age group* L_a	*Death rate per 1 000* m_a
0,5	Under 1	366 651	366 651	100 000	36 665	72 501	505.71
3.0	1-4	150 864	517 515	63 335	15 337	221 658	69.19
7.5	5-9	38 967	556 482	48 248	3 896	231 500	16.83
12.5	10-14	24 421	580 903	44 352	2 442	215 655	11.32
17.5	15-19	30 560	611 463	41 910	3 056	201 910	15.14
22.5	20-24	34 964	646 427	38 854	3 497	185 528	18.85
27.5	25-29	31 494	677 921	35 357	3 149	168 913	18.64
32.5	30-34	27 691	705 612	32 208	2 769	154 118	17.97
37.5	35-39	24 621	730 233	29 439	2 462	141 040	17.56
42.5	40-44	22 820	753 053	26 977	2 282	129 180	17.66
47.5	45-49	23 153	776 206	24 695	2 316	117 685	19.68
52.5	50-54	24 755	800 961	22 379	2 475	105 708	23.41
57.5	55-59	26 890	827 851	19 904	2 689	92 798	28.98
62.5	60-64	30 760	858 611	17 215	3 076	78 385	39.24
67.5	65-69	34 430	893 041	14 139	3 443	62 088	55.45
72.5	70-74	36 231	929 272	10 696	3 623	44 423	81.56
77.5	75-79	32 495	961 767	7 073	3 250	27 240	119.31
82.5	80-84	22 686	984 453	3 823	2 268	13 445	168.69
87.5	85 +	15 547	1 000 000	1 555	1 555	4 963	313.32

THIRD EXAMPLE: THE CRUDE DEATH RATE d_0 IS GIVEN

The crude death rate is equal to the difference between the crude birth rate and the rate of natural increase. Thus, it is a function of r. On graph VI.1 the curve $d(r) = b(r) - r$ is easily traced.

The straight line of the ordinate d_0 cuts this curve at a point M_0 whose abscissa is the rate of natural increase r_0 of the population. There is one population, and only one which answers the question.

FOURTH EXAMPLE: THE AGE DISTRIBUTION OF THE POPULATION AT A GIVEN AGE $C_0(_0)$ IS KNOWN

The method in this case is exactly the same. For a given age, $C_0(a_0)$ is a function of r. Table VI.5 enables these functions to be plotted for various age groups. It may be seen in graph VI.2 that above the 25-29 age group the curve passes through a minimum, while below that age group it passes through a maximum. The abscissae of the points of intersection of the straight

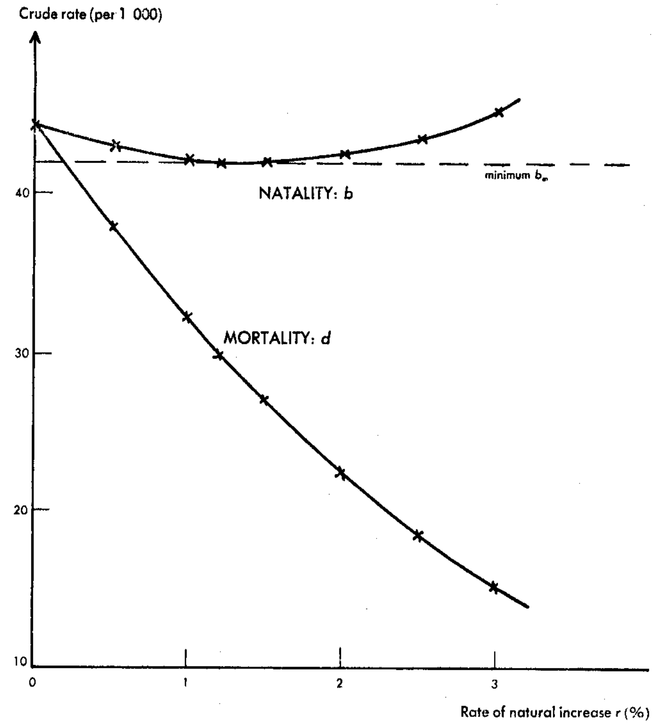
TABLE VI.4. FEMALE SURVIVORSHIP FUNCTION OF MALTHUSIAN POPULATIONS OF THE SUB-SET $G_0(r)$ LINKED TO THE AGE DISTRIBUTION OF FEMALE DEATHS GIVEN IN TABLES VI.1, VI.2 AND VI.3, FOR THE THREE RATES OF VARIATION: $r = 0.000$ (STATIONARY POPULATION), $r = 0.015$ AND $r = 0.030$

Age	$r = 0.000$ (a)	$r = 0.015$ (b)	$r = 0.030$ (c)
0	100 000	100 000	100 000
1	63 335	76 231	87 628
5	48 248	66 078	82 141
10	44 352	63 272	80 519
15	41 910	61 376	79 337
20	38 854	58 818	77 620
25	35 357	55 665	75 337
30	32 208	52 604	72 949
35	29 439	49 702	70 508
40	26 977	46 922	67 987
45	24 695	44 144	65 273
50	22 379	41 106	62 073
55	19 904	37 604	58 097
60	17 215	33 505	53 080
65	14 139	28 450	46 413
70	10 696	22 352	37 742
75	7 073	15 436	27 141
80	3 823	8 750	16 094
85	1 555	3 718	7 134

(a) Figure taken from the fifth column of table VI.3.
(b) Figure taken from the eighth column of table VI.2.
(c) Figure taken from the eighth column of table VI.1.

TABLE VI.5. DISTRIBUTION BY AGE GROUPS OF WOMEN IN MALTHUSIAN POPULATIONS OF THE SUB-SET $G_0(r)$, LINKED TO THE AGE DISTRIBUTION OF FEMALE DEATHS GIVEN IN TABLES VI.1, VI.2 and VI.3, FOR THE THREE RATES OF VARIATION: $r = 0.000$ (STATIONARY POPULATION), $r = 0.015$ and $r = 0.030$.

	$r = 0.000$	$r = 0.015$	$r = 0.030$
Crude birth rate per 100 000 . . .	44.070	42.165	45.070
Age group	*Age distribution*		
0	3 196	3 462	4 028
1-4	9 770	11 438	13 964
5-9	10 205	12 186	14 636
10-14	9 506	10 896	12 380
15-19	8 900	9 744	10 462
20-24	8 178	8 615	8 775
25-29	7 446	7 557	7 322
30-34	6 793	6 621	6 097
35-39	6 215	5 803	5 066
40-44	5 693	5 075	4 196
45-49	5 186	4 485	3 451
50-54	4 660	3 773	2 803
55-59	4 090	3 163	2 232
60-64	3 455	2 558	1 719
65-69	2 736	1 945	1 252
70-74	1 950	1 343	831
75-79	1 209	798	476
80-84	0 593	381	220
85 and over	0 219	151	90
ALL AGES	100 000	100 000	100 000



Graph VI.1. Variations, as a function of r, in the crude birth rate and crude death rate of Malthusian populations of the set G(r), linked to the age distribution of deaths d_a in tables VI.1, VI.2 and VI.3 (the distribution is the same in all three tables)

line of the ordinate $C_0(a_0)$ with the corresponding curve are the rates of natural increase of the populations sought. As in the first example, there is not always a solution, and when there is one solution there is generally a second one also.[1]

FIFTH EXAMPLE: THE FERTILITY FUNCTION (a) IS GIVEN

In the populations sought we have the relation:

$$\int_u^v e^{-ra} p_f(a) \varphi_f(a) da = 1$$

Let us consider the integral

$$I(r) = \int_u^v e^{-ra} p_f(a) \varphi_f(a) da$$

This can be calculated very easily with the aid of tables VI.1, VI.2 and VI.3 for the values of r assumed to be $r = 0$, $r = 0.015$ and $r = 0.03$.

Table VI.6 gives the computation for a fertility function corresponding to a gross reproduction rate of 2.9 according to the intermediate model fertility distribution. On graph VI.3 we have traced the curve representing I(r) as a function of r. In the graph the straight line of the ordinate 1 cuts the curve at two points M_1 and M_2 whose abscissae r_1 and r_2 define two populations satisfying the given condition, i.e., having the specific fertility defined above.

As has been seen, there is not always a solution to the problem posed. Depending on the values of $\varphi(a)$, it may be that the curve of graph VI.3 is entirely above the straight line of the ordinate 1. We have also seen that, when there is one solution, there is generally a second one also.

This fifth example is similar to the second example of the sub-sets H(r), which was defined by knowledge of the survivorship function $p(a)$ and the additional knowledge of the fertility function $\varphi(a)$. This was shown to be a particular Malthusian population to which Lotka gave the name of a stable Malthusian population or, more simply, a stable population.

Here, in a sub-set G(r) defined by a constant age distribution of deaths, the additional knowledge of the fertility function defines, if certain conditions are satisfied, two particular Malthusian populations.

We shall confine ourselves to these few examples, although others could easily be imagined. We shall now proceed to a study of processes of demographic evolution where there is a constant age distribution of deaths.

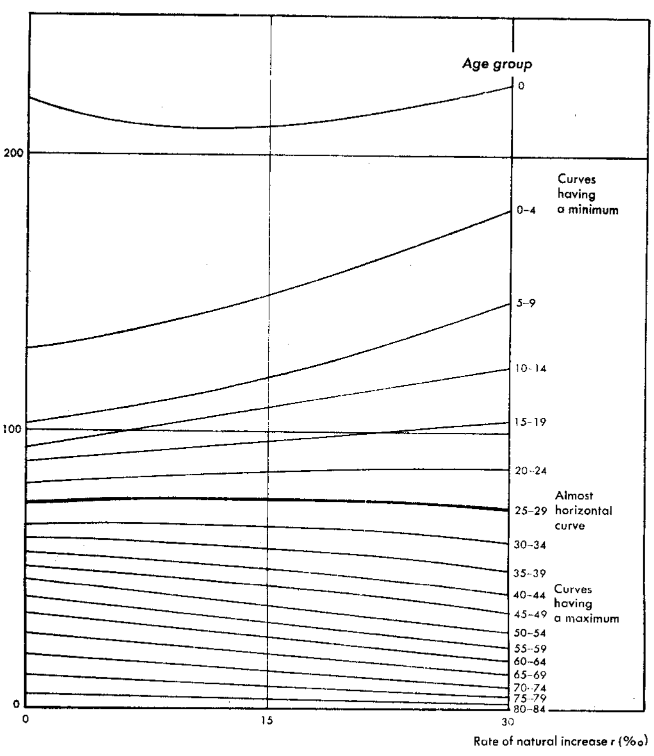
B. Processes of demographic evolution where there is constant age distribution of deaths

As in the case of the sub-sets H(r) and F(r), we can try to establish a correspondence between the sub-sets G(r) and processes of demographic evolution starting from a given initial state and assuming a constant age distribution of deaths associated with another condition implied in the series of examples considered above.

It is easy to see in a particular case that such processes of population evolution are not defined by these two conditions. Let us imagine, for example, that we assume the following two conditions:

(i) Constant age distribution of deaths $d_0(a)$; and

(ii) Constant crude death rate d_0.



Graph VI.2. Graphic illustration of variations, as a function of r, in the age distribution of women of Malthusian populations of the sub-set $G_0(r)$, linked to the age distribution of female deaths d_a in tables VI.1, VI.2 and VI.3 (the distribution is the same in all three tables)

[1] There is only one solution if the straight line of the ordinate $C_0(a_0)$ is tangent to the curve corresponding to age a_0.

TABLE VI.6. COMPUTATION OF THE INTEGRAL: $I(r) = \int_u^v e^{-ra} p_f(a) \varphi_f(a) da$ IN THREE MALTHUSIAN FEMALE POPULATIONS OF THE SUB-SET $G_0(r)$, LINKED TO AGE DISTRIBUTION OF FEMALE DEATHS IN TABLES VI.1, VI.2 AND VI.3, CORRESPONDING TO THREE RATES OF NATURAL VARIATION: $e = 0{,}030$, $r = 0.015$ AND $r = 0.000$

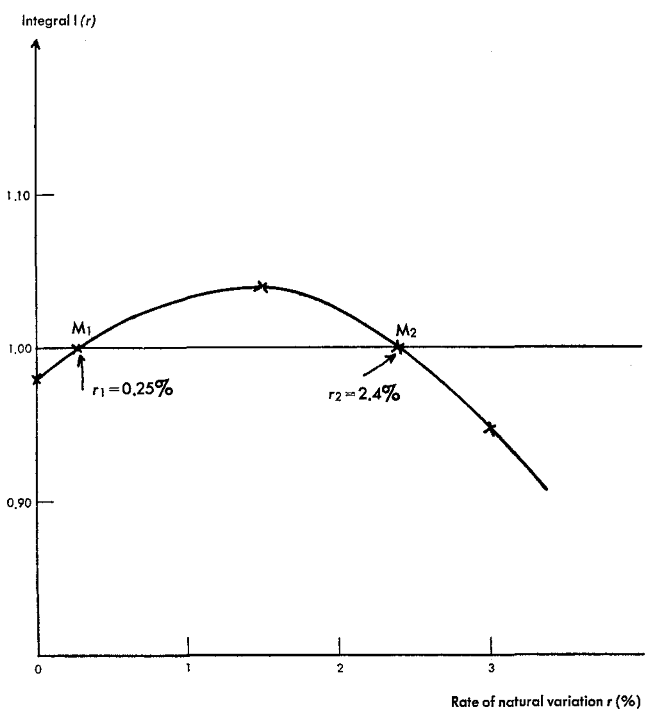
The gross reproduction rate is 2.9 and the distribution of the female fertility rates is that of the intermediate model

Median age a	Age group (years) a	Age distribution of female fertility rates	$r = 0.000$: $e^{-ra}L_a$ [a]	$r = 0.000$: Product of the two preceding columns	$r = 0.015$: $e^{-ra}L_a$ [b]	$r = 0.015$: Product of preceding column and third column	$r = 0.030$: $e^{-ra}L_a$ [c]	$r = 0.030$: Product of preceding column and third column
17.5	15-19	0.100	201 910	20 191	230 920	23 092	232 110	23 211
22.5	20-24	0.273	185 528	50 650	204 270	55 770	194 686	53 150
27.5	25-29	0.263	168 913	44 440	179 200	47 130	162 447	42 730
32.5	30-34	0.188	154 118	28 980	157 080	29 520	135 260	25 430
37.5	35-39	0.121	141 040	17 060	137 620	16 650	112 394	13 600
42.5	40-44	0.055	129 080	7 105	120 350	6 620	93 083	5 120
	15-44 years	1.000		168 426		178 782		163 241
	$I(r)$ [d]			0.977		1.037		0.9469

[a] Figures from the seventh column of table IV.3.
[b] Figures from the penultimate column of table IV.2, multiplied by e^{-ra}
[c] Figures from the fifth column of table II.3.
[d] Figures from the preceding line, multiplied by (2.9/5.0) (1/100 000). The division by 100 000 is necessary because the life tables are expressed on the basis of an initial number at birth equal to 100 000.

These two conditions correspond to the third example given above.

Starting from a given initial state at time t, the two conditions enable us to determine at each age the survivors at time $t + dt$ of the persons living at time t. They do not, however, enable us to determine the survivors at time $t + dt$ of persons born between times t and $t + dt$. In order to calculate these survivors we need a third condition, such as the fertility function. Let us then add this third condition. We can then compute a population projection, starting from the initial time. Let us consider what would happen to the population if we continued the computation of the projection indefinitely.



Graph VI.3. Graphic presentation of the integral

$$\int_u^v e^{-ra} p_f(a) \varphi_f(a) da$$

of Malthusian female populations of the sub-set $G_0(r)$, linked to the age distribution of female deaths in tables VI.I, VI.2 and VI.3 (the distribution is the same in all three tables), the gross reproduction rate being 2.9 and the age distribution of the female fertility rates being the intermediate model fertility distribution

It is easy to see that, generally, the characteristics of the population computed in this way will not approach any limit, but will continue to fluctuate indefinitely without any diminution of the fluctuations with the passage of time.

In fact, if a limit existed—if, for example, the age distribution of the population tended to become invariable—this would mean that at the limit we should find a Malthusian population of the sub-set $G_0(r)$ linked to the age distribution of deaths $d_0(a)$. In such a sub-set, knowledge of the crude death rate defines a particular Malthusian population whose rate of natural increase is r, computed as described in the third example above. This rate r_0 would be that of the limit population.

In the sub-set $G_0(r)$, however, knowledge of the fertility function $\varphi(a)$ determines, subject to certain conditions, two particular Malthusian populations whose rate of increase r_1 and r_2 are computed as described in the fifth example above, and the rate of natural increase of the imagined limit population must be one of these two values r_1 and r_2.

Generally, r_0 will be different from r_1 and r_2, and therefore the limit population envisaged cannot exist. It is only when the fertility function $\varphi(a)$ is precisely so selected that one of these values r_1 or r_2 is equal to r_0 that a limit population can exist. It would still remain to be seen, however, whether the limit population really existed, since the fact that it can exist does not necessarily mean that it does exist.

To sum up, processes of demographic evolution where the age distribution of deaths is constant do not generally lead to stable situations, as was the case with processes of demographic evolution where the survivorship function was constant.

We shall confine ourselves to these remarks and shall now proceed, in the next two chapters, to a study of quasi-stable populations, which, it may be recalled, represent an approximation to stable populations.