Probabilistic Projections of the Total Fertility Rate

Leontine Alkema\textsuperscript{1},
Adrian Raftery\textsuperscript{2}, Patrick Gerland\textsuperscript{3}, Sam Clark\textsuperscript{2},
François Pelletier\textsuperscript{3} and Thomas Buettner\textsuperscript{3}

\textsuperscript{1}National University of Singapore, Singapore,
\textsuperscript{2}University of Washington, Seattle,
\textsuperscript{3}United Nations Population Division, New York

Funded by NICHD grant number 1 R01 HD054511 01 A1
Introduction

TFR time series since 1950 can be described with 3 phases:

1. Pre-transition high fertility
2. Fertility transition
3. Post-transition low fertility

We modeled the 5-year changes in the TFR in Phase II and III, using UN estimates 1950-2010. The observation period is split into the different phases:

Start of Phase II is before 1950 (if max TFR is below 5.5 children), or at latest local max. within 0.5 child of global max.

→ All countries are currently in phase II or III
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UN (Dec 4, 2009) Probabilistic TFR projections
Fertility transition

Start of Phase III

Post-transition low fertility

Results

Formal definition is based on model parameters (later in presentation)

Within observation period:

Start of phase III is approximated by the midpoint of earliest two subsequent increases below 2

Start of phase III before 2005-2010 observed in 20 countries

(Singapore, Bulgaria, Czech Republic, Russian Federation, Channel Islands, Denmark, Estonia, Finland, Latvia, Norway, Sweden, United Kingdom, Italy, Spain, Belgium, France, Germany, Luxembourg, Netherlands, United States of America)

<table>
<thead>
<tr>
<th>Year</th>
<th>TFR</th>
<th>Phase II</th>
<th>Phase III</th>
</tr>
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<tbody>
<tr>
<td>1960</td>
<td></td>
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<td></td>
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<tr>
<td>1970</td>
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<td>1980</td>
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<tr>
<td>1990</td>
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<td></td>
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<tr>
<td>2000</td>
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</tr>
</tbody>
</table>

1.0 1.5 2.0 2.5

Bulgaria

Period

UN (Dec 4, 2009)

Probabilistic TFR projections
Start of Phase III

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Outline

1. Phase II: The fertility transition
2. Phase III: Post-transition low fertility
3. Results
Outline

1 Phase II: The fertility transition

2 Phase III: Post-transition low fertility

3 Results
Current UN projections for high fertility countries
Current UN projections for high fertility countries

UN estimates for Burkina Faso:

- 1960: TFR
- 1980: TFR
- 2000: 5.8
- 2020: 5.8
- 2040: 4.7

UN projects TFR will decrease to 1.85 children/woman

Deterministic projection model:

\[ f_{c,t+1} = f_{c,t} - d(\theta, f_{c,t}) \]

- \( f_{c,t} \): TFR for country \( c \), 5-year period
- \( d(\theta, f_{c,t}) \): 5-year decline given by decline function
- \( \theta \): parameter vector chosen from \( \{ \theta_{SS}, \theta_{FS}, \theta_{FF} \} \)
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- Deterministic model: No uncertainty assessment
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2. Related to $\theta \in \{\theta_{SS}, \theta_{FS}, \theta_{FF}\}$:
   - 5-year decrements are not country-specific
   - 3 sets of parameter values do not capture the variation in past
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Extend the UN model

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1. Estimate \( \theta \) in \( d(\theta, f_{c,t}) \) for each country
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   - Allow for random distortions
   - Assess uncertainty in \( \theta_c \)

\[ f_{c,t} \text{ for country } c, 5\text{-year period} \]
\[ d(\theta, f_{c,t}) 5\text{-year decrement} \]
\( \varepsilon_{c,t} \) Random distortions
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Random walk with drift:

\[
f_{c,t+1} = f_{c,t} - d(\theta_c, f_{c,t}) + \varepsilon_{c,t},
\]

with

\[
\begin{align*}
&\begin{cases}
  f_{c,t} & \text{TFR for country } c, \text{ 5-year period } t \\
  d(\theta_c, f_{c,t}) & \text{5-year decrement} \\
  \varepsilon_{c,t} & \text{Random distortions}
\end{cases}
\end{align*}
\]
5-year decrements
5-year decrements

\[ d(\theta_c, f_c, t) = d_c \left( \frac{\frac{\ln(81)}{\Delta c_1} (f_c t - \sum_i \Delta c_i + 0.5 \Delta c_1)}{1 + \exp\left(-\ln(81) (f_c t - \sum_i \Delta c_i + 0.5 \Delta c_1)\right)} \right) + \frac{\frac{1}{\ln(81) (f_c t - \Delta c_4 - 0.5 \Delta c_3)}{1 + \exp\left(-\ln(81) (f_c t - \Delta c_4 - 0.5 \Delta c_3)\right)} \right) \]
5-year decrements

\[ d(\theta_c, f_{c,t}) = d_c \left( \frac{-1}{1+\exp\left(-\frac{\ln(81)}{\Delta c_1}(f_{ct} - \sum_i \Delta c_i + 0.5 \Delta c_1)\right)} + \frac{1}{1+\exp\left(-\frac{\ln(81)}{\Delta c_3}(f_{ct} - \Delta c_4 - 0.5 \Delta c_3)\right)} \right) \]
5-year decrements

\[ d(\theta_c, f_c, t) = d_c \left( \frac{-1}{1 + \exp\left( -\ln(81) (f_{ct} - \sum_i \Delta c_i + 0.5 \Delta c_1) \right)} + \frac{1}{1 + \exp\left( -\ln(81) (f_{ct} - \Delta c_4 - 0.5 \Delta c_3) \right)} \right) \]

\[ \theta_c = (\Delta c_1, \Delta c_2, \Delta c_3, \Delta c_4, d_c) \]
5-year decrements

\[ d(\theta_c, f_{c,t}) = d_c \left( \frac{-1}{1 + \exp(-\frac{\ln(81)}{\Delta_c1} (f_{ct} - \sum_i \Delta_{ci} + 0.5 \Delta_{c1}))} + \frac{1}{1 + \exp(-\frac{\ln(81)}{\Delta_c3} (f_{ct} - \Delta_{c4} - 0.5 \Delta_{c3}))} \right) \]

- \( \theta_c = (\Delta_{c1}, \Delta_{c2}, \Delta_{c3}, \Delta_{c4}, d_c) \)
- Start level \( U_c = \sum_i \Delta_{ci} \) is observed, or estimated if decline started < 1950,
5-year decrements

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- Other parameters:
  \( d_c, \Delta c_4 \), and for \( i = 1, 2, 3 \)
  proportions \( p_{ci} = \frac{\Delta c_i}{\sum_{j=1}^{3} \Delta c_j} \)

\( f_c(t) \) (decreasing)
5-year decrements

d(\theta_c, f_{c,t}) = d_c \left( \frac{\exp\left(-\frac{\ln(81)}{\Delta c_1} (f_{ct} - \sum_i \Delta c_i + 0.5 \Delta c_1)\right)}{1+\exp\left(-\frac{\ln(81)}{\Delta c_3} (f_{ct} - \Delta c_4 - 0.5 \Delta c_3)\right)} + \frac{1}{1+\exp\left(-\frac{\ln(81)}{\Delta c_3} (f_{ct} - \Delta c_4 - 0.5 \Delta c_3)\right)} \right)

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- Estimate these parameters with a Bayesian hierarchical model
Bayesian hierarchical model

Bayesian inference: unknown parameters have probability distributions, which are "updated" with new information (prior distribution + data and model → posterior distribution).

Exchange information between countries using a hierarchical model: Unknown decline parameters are distributed around a "world average". For a specific country, its parameters estimates are determined by its observed declines, as well as the world level experience.

Example: maximum 5-year decrement $d^c$ to restrict it to between 0.25 and 2.5 child:

\[ d^c = \log( d^c - 0.25 ) - \log( 2.5 - d^c ) \]

Assume that $d^c$'s are exchangeable between countries: $d^c \sim N(\chi,\psi^2)$, with $\chi$ the world mean, and $\psi^2$ the variance of the $d^c$'s.
Bayesian hierarchical model

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Bayesian hierarchical model

The model is given by:

\[ f_{c,t+1} = f_{c,t} - d(\theta_c, f_{c,t}) + \epsilon_{c,t} \]

Hierarchical distributions for country-specific parameters

Prior distributions on the hierarchical parameters, and variance parameters of the distortion terms

Use Markov Chain Monte Carlo (MCMC) algorithm to get many samples of the set of model parameters

Each set of model parameters gives a future TFR trajectory

Many sets → Many TFR trajectories → Median projection and projection intervals
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Hierarchical distributions for country-specific parameters $\theta_c$
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- Many sets
  - Many TFR trajectories
  - Median projection and projection intervals
How to construct a future TFR trajectory?

To get future $f_{c, t+1}$ for country $c$ in Phase II:

1. Outcome of $\theta_c$ gives decrement $d(\theta_c, f_{c, t})$.
2. Sample a distortion $\epsilon_{c, t}$ (use outcomes of its variance parameters).
3. $f_{c, t+1} = f_{c, t} - d(\theta_c, f_{c, t}) + \epsilon_{c, t}$.

Repeat until start of Phase III:

- earliest $t$ such that $\min \{f_{c, s}, s = 1, \ldots, t\} \leq \Delta_c$,
- and $f_{c, t} > f_{c, t-1}$.
How to construct a future TFR trajectory?

- To get future $f_{c,t+1}$ for country $c$ in Phase II:

  \[ f_{c,t} + 1 = f_{c,t} - d(\theta_c, f_{c,t}) + \epsilon_{c,t} \]

  Repeat until start of Phase III:

  earliest $t$ such that $\min\{f_{c,s}, s=1, ..., t\} \leq \triangle c^4$, AND $f_{c,t} > f_{c,t-1}$
How to construct a future TFR trajectory?

- To get future $f_{c,t+1}$ for country $c$ in Phase II:

$$f_{c,t+1} = f_{c,t} - d(\theta_c, f_c,t) + \varepsilon_{c,t}$$

Repeat until start of Phase III:

$$\text{earliest } t \text{ such that } \min\{f_{c,s} : s = 1, \ldots, t\} \leq \Delta_c$$

and $f_{c,t} > f_{c,t-1}$.
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![Graph showing TFR data and projections for India](image-url)
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  - $f_{c,t+1} = f_{c,t} - d(\theta_c, f_{c,t}) + \varepsilon_{c,t}$

- Repeat until start of Phase III:
  earliest $t$ such that $\min\{f_{c,s} : s = 1, \ldots, t\} \leq \triangle c_{4}$, AND $f_{c,t} > f_{c,t-1}$

\[\begin{array}{cccccccccccccccc}
\text{TFR} & 0.5 & 1.0 & 1.5 & 2.0 & 2.5 & 3.0 & 3.5 \\
\text{1993} & & & & & & & \\
\text{1998} & & & & & & & \\
\text{2003} & & & & & & & \\
\text{2008} & & & & & \bullet & & \\
\text{2013} & & & & & & \bullet & & \\
\text{2018} & & & & & & & \bullet & & \\
\text{2023} & & & & & \bullet & & & \bullet & & \\
\text{2028} & & & & & & & & \bullet & & & \bullet &
\end{array}\]

\[\begin{array}{cccccccccccccccc}
\text{5-year decrement} & 0.0 & 0.2 & 0.4 & 0.6 & 0.8 & 1.0 \\
\text{1993} & & & & & & & \\
\text{1998} & & & & & & & \\
\text{2003} & & & & & & & \\
\text{2008} & & & & & & & \\
\text{2013} & & & & & & & \\
\text{2018} & & & & & & & \\
\text{2023} & & & & & & & \\
\text{2028} & & & & & & & \\
\text{2033} & & & & & & & \\
\text{2038} & & & & & & & \\
\text{2043} & & & & & & & \\
\text{2048} & & & & & & & \\
\text{2053} & & & & & & & \\
\text{2058} & & & & & & & \\
\text{2063} & & & & & & & \\
\text{2068} & & & & & & & \\
\text{2073} & & & & & & & \\
\text{2078} & & & & & & & \\
\text{2083} & & & & & & & \\
\text{2088} & & & & & & & \\
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\text{2098} & & & & & & & \\
\end{array}\]

- UN estimate

- Trajectory

- Decline curve

- Decrments
How to construct a future TFR trajectory?

- To get future $f_{c,t+1}$ for country $c$ in Phase II:
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# Outline

1. **Phase II: The fertility transition**

2. **Phase III: Post-transition low fertility**

3. **Results**
Phase III: What happens post-fertility-transition?

Assume TFR will fluctuate around 2.1, use an AR(1) model:

$$f_t = f_{t-1} + (1 - \rho)(2.1 - f_{t-1}) + \epsilon_t$$

$$\epsilon_t \sim N(0, \sigma^2)$$
Phase III: What happens post-fertility-transition?

$$f(t) = f(t-1)$$

UN: $$f(t) = f(t-1) + 0.05$$

Assume TFR will fluctuate around 2.1, use an AR(1) model:

$$f_t = f_{t-1} + (1 - \rho)(2.1 - f_{t-1}) + e_t$$

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# Fertility transition

## Post-transition low fertility

<table>
<thead>
<tr>
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<th>TFR</th>
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<tbody>
<tr>
<td>2010</td>
<td>2.1</td>
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<tr>
<td>2035</td>
<td>1.85</td>
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<td>2060</td>
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<tr>
<td>2110</td>
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</tr>
<tr>
<td>2135</td>
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</table>

Asymptotic 95% projection interval given by [1.7, 2.5]

---

UN (Dec 4, 2009)

Probabilistic TFR projections
TFR projection that starts at 1.5 in 2005-2010

AR(1) simulations

TFR = 2.1
TFR = 1.85

Asymptotic 95% projection interval (PI) given by [1.7, 2.5]
TFR projection that starts at 1.5 in 2005-2010

AR(1) simulations

Period

TFR

2010 2035 2060 2085 2110 2135

● Median

95% PI

Trajectories

TFR = 2.1

TFR = 1.85

Asymptotic 95% projection interval (PI) given by [1.7, 2.5]

UN (Dec 4, 2009)

Probabilistic TFR projections
TFR projection that starts at 1.5 in 2005-2010

Asymptotic 95% projection interval (PI) given by [1.7,2.5]
Fertility transition

Post-transition low fertility

Results

Increased uncertainty in long range Phase III projections

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</tr>
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<tbody>
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</table>

AR(1) simulations

Probabilistic TFR projections
Increased uncertainty in long range Phase III projections

- Use all below-replacement TFRs to estimate uncertainty in long-term projections
  \( s^{(a)} = 0.203 \) after 4 periods in phase III)
Increased uncertainty in long range Phase III projections

- Use all below-replacement TFRs to estimate uncertainty in long-term projections
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- In far future, the 95% projection interval is given by \([1.2,3.0]\)
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  \( s^{(a)} = 0.203 \) after 4 periods in phase III
- In far future, the 95% projection interval is given by \([1.2, 3.0]\)
Outline

1. Phase II: The fertility transition
2. Phase III: Post-transition low fertility
3. Results
### Fertility transition

<table>
<thead>
<tr>
<th>Country</th>
<th>Period</th>
<th>TFR 1993</th>
<th>TFR 2008</th>
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<th>TFR 2068</th>
<th>TFR 2083</th>
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</table>

UN (Dec 4, 2009) Probabilistic TFR projections 18 / 23
Projections

Burkina Faso

- Median projection
- Median $\pm$ 0.5 child
- 80% PI
- 95% PI

Period
- TFR
- 1993
- 2008
- 2023
- 2038
- 2053
- 2068
- 2083
- 2098

UN estimates
UN projection

UN (Dec 4, 2009)
Probabilistic TFR projections
Projections

Burkina Faso

UN estimates
UN projection
Median projection
Median +/- 0.5 child
80% PI
95% PI

Period
TFR
1993 2008 2023 2038 2053 2068 2083 2098

Italy

UN estimates
UN projection
Median projection
Median +/- 0.5 child
80% PI
95% PI

Period
TFR
1993 2008 2023 2038 2053 2068 2083 2098

UN (Dec 4, 2009)
### Decline curve; world level

<table>
<thead>
<tr>
<th>Fertility transition</th>
<th>Post-transition low fertility</th>
<th>Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>Decline curve</td>
<td>world level</td>
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</tr>
</tbody>
</table>

_UN (Dec 4, 2009)

Probabilistic TFR projections

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Decline curve; world level
Decline curve; world level

Decoments
- UN declines
- Hierarchical mean

TFR (reversed)

5-year decrement

UN (Dec 4, 2009)

Probabilistic TFR projections
Decline curve; country-specific
Decline curve; country-specific

Thailand

- 5-year decrements
- Median
- 95% PI
- Example curves

UN declines

TFR (reversed)

TFR decrement

Median

Example curves

UN declines

5-year decrements

UN (Dec 4, 2009)

Probabilistic TFR projections
Decline curve; country-specific

**Thailand**

- **5-year decrements**
- **UN declines**
- **Median**
- **95% PI**
- **Example curves**

**India**

- **5-year decrements**
- **UN declines**
- **Median**
- **95% PI**
- **Example curves**
Decline curve; Burkina Faso

Burkina Faso

5-year decrements

UN declines

Median

95% PI

Example curves

TFR decrement

TFR (reversed)

Probabilistic TFR projections

UN (Dec 4, 2009) | 21 / 23
Out-of-sample model validation

<table>
<thead>
<tr>
<th>Country</th>
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<td>India</td>
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<td>2003</td>
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<td>1963</td>
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<td>1973</td>
<td>7</td>
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Summary of model validation results:
- Project Above Coverage Median 95%PI 80%PI
  - from 1980: 43% 91% 77%
  - from 1995: 36% 93% 79%

UN (Dec 4, 2009)

Probabilistic TFR projections
Out-of-sample model validation

- Use data until 1980, and project until 2005-2010:
Out-of-sample model validation

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Out-of-sample model validation

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![Graphs showing TFR projections for India and Thailand](image-url)
Out-of-sample model validation

- Use data until 1980, and project until 2005-2010:

  India
  
  Median projection
  95% PI

  Thailand
  
  Median projection
  95% PI

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Summary Bayesian TFR Projection Model

During the fertility transition:
- The 5-year decreases are modeled as a function of TFR level and decline parameters, with random distortions added to it.
- The decline parameters are estimated with a Bayesian hierarchical model.

After the fertility transition the TFR will converge to/fluctuate around 2.1, using an AR(1) model.

Results:
- Country-specific projections that include an uncertainty assessment.
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- Probabilistic projection model for 5-year changes during and after the fertility transition
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