Chapter V

PROCESSES OF DEMOGRAPHIC EVOLUTION WHERE THE AGE DISTRIBUTION IS INVARIABLE: SEMI-MALTHUSIAN POPULATIONS

A. Introduction

The populations considered in chapters II and IV were Malthusian populations, i.e., populations with constant mortality and age distributions.

In chapter II it was further assumed that mortality was known, and this additional condition defined what we termed the sub-sets H(r). The age distribution remained invariable but was not assumed to be known.

In chapter IV it was assumed that the age distribution was known, and this additional condition defined what we termed the sub-sets F(r). The mortality in these sub-sets remained invariable but was not assumed to be known.

In each of those sub-sets, knowledge of an additional condition generally determined a particular Malthusian population (or a small number of particular Malthusian populations). Ultimately, each particular population was determined by three conditions which were, for the two types of sub-sets, as follows:

Sub-sets H(r)

(a) Mortality invariable and known;

(b) Age distribution invariable but not known;

(c) Knowledge of one other characteristic of the population, e.g.:

(i) Rate of natural variation;
(ii) Crude birth rate;
(iii) Crude death rate;
(iv) Age distribution at a given age;
(v) Age-specific fertility ("stable" Malthusian population).

Sub-sets F(r)

(a) Mortality invariable but not known;

(b) Age distribution invariable and known;

(c) Knowledge of one other characteristic of the population, e.g.:

(i) Rate of natural variation;
(ii) Crude death rate;
(iii) Age distribution of deaths at a given age;
(iv) Survivorship function at a given age.

In chapter III we defined some processes of demographic evolution where the condition that the *age distribution was invariable but not known* was eliminated from the conditions of sub-set H(r).

We first considered the possibility of the existence of such processes and then went on to study their properties.

We concluded that in such processes the population generally developed towards the Malthusian population (or one of the Malthusian populations) defined by restoring the condition of invariability of the age distribution.

In the case of the sub-sets F(r) we may pose the same question as in the case of the sub-sets H(r), and we can study the processes of demographic evolution defined by eliminating from the conditions of the sub-set F(r) the condition that *the mortality is invariable but not known*.

This is the subject of the present chapter.

B. Processes where the age structure is invariable

Definition of semi-Malthusian populations

Let us consider a population at time t, defined by N(t) as the total population, $r(t)$ the rate of natural variation, $p(a, t)$ the survivorship function, and C(a) the age distribution assumed to be independent of time.

Let us now see what happens to this population at time $t + dt$. The number of persons aged a at time t is equal to N(t)C(a). At time $t + dt$, the survivors are aged $a + dt$ and their number is equal to:

$$N(t + dt)C(a + dt) = N(t)C(a + dt)[1 + r(t)dt]$$

We can also estimate the number of survivors by applying the survivorship function. The number of survivors is:

$$N(t)C(a) \frac{p(a + dt, t)}{p(a, t)}$$

The two expressions obtained for the number of persons surviving to time $t + dt$, give:

$$N(t)C(a + dt)\,[1 + r(t)dt] = N(t)C(a) \frac{p(a + dt, t)}{p(a, t)}$$

which is written:

$$\frac{C(a + dt)}{C(a)} [1 + r(t)dt] = \frac{p(a + dt, t)}{p(a, t)}$$

or:

$$\left[1 + \frac{C'(a)}{C(a)} dt\right] [1 + r(t)dt] = 1 + \frac{p'_a(a, t)}{p(a, t)} dt$$

and if we disregard the second-order terms, we obtain:

$$1 + \frac{C'(a)}{C(a)} dt + r(t)dt = 1 + \frac{p'_a(a, t)}{p(a, t)} dt$$

and finally,

$$\frac{C'(a)}{C(a)} = \frac{p'_a(a,t)}{p(a,t)} - r(t)$$

By integration, we obtain:

$$C(a) = b_0 e^{-r(t)a} p(a,t) \qquad \text{(V.1)}$$

where b_0 is a constant equal to C(0), the crude birth rate. Formula V.1 is the fundamental formula for Malthusian populations. We can therefore state the following result:

In a process of demographic evolution where the age structure of the population is constant, there are, at all times, the same relations between the demographic characteristics of the population as in a Malthusian population.

We shall term populations with a constant age distribution "*semi-Malthusian populations*".

Another definition of semi-Malthusian populations

Let us assume that, from a given time onwards, the mortality and fertility of a population remain invariable at the level they have reached at that moment. It was seen in chapter III that this led to a process of population evolution in which the population approached the stable Malthusian population corresponding to the mortality and fertility of the moment. Generally, the limit age distribution is different from the age distribution at the time when the process began. It should be noted that we say "generally", because, as has been seen, in a population which is assumed to maintain an invariable age distribution, the age distribution of the limit stable population coincides exactly at all times with the current age distribution.

In other words, in order for the current age distribution to coincide at all times with the age distribution of the limit stable population, it is sufficient that the population should have an invariable age structure.

It is easy to show that this condition is *not only sufficient, but also necessary*. The fact that at time t the age distribution $C_f(a, t)$ coincides with the age distribution according to the laws of $p_f(a, t)$ and $\varphi_f(a, t)$ means that, if we assume that from time t onwards the mortality and fertility functions retain the values they had reached at time t, the population will approach a stable state whose age distribution will indeed be $C_f(a, t)$. Thus we hall have:

$$p(a,t) = \frac{C(a,t)}{C(0,t)} e^{r(t)a}$$

In such a life table we have:

$$\frac{p(a+da,t)}{p(a,t)} = \frac{C(a+da,t)}{C(a,t)} e^{r(t)da}$$

At time $t + dt$ the survivors of the persons aged a at time t will be aged $a + dt$ and will number:

$$N(t)C(a,t)\frac{p(a+dt,t)}{p(a,t)} =$$

$$= N(t)C(a,t)\frac{C(a+dt,t)}{C(a,t)} e^{r(t)dt} = N(t)C(a+dt,t)e^{r(t)dt}$$

However, they will also number:

$$N(t)e^{r(t)dt}C(a+dt, t+dt)$$

whence we obtain by equating the two expressions:

$$C(a+dt,t) = C(a+dt,t+dt)$$

In other words, at time $t + dt$ the age distribution is the same as at time t. The age distribution is therefore invariable. *Thus, semi-Malthusian populations are identical with populations for which the current age structure coincides with the stable age structure.*

Relationship between semi-Malthusian populations and quasi-stable populations

In a previous study,[1] quasi-stable populations were defined as populations with constant fertility where the mortality varied within the universe of the intermediate series of model life tables. It was noted that such populations had current age structures which almost coincided with their stable structures. *Quasi-stable populations thus offer an example of the near-achievement of semi-stable populations. We shall revert to this point in chapters VII and VIII.*

Fertility in semi-Malthusian populations

The fertility function must verify the relation:

$$\int_u^v C(a)\varphi(a,t)da = C(0) = b_0 \qquad \text{(V.2)}$$

There is no strictly mathematical reason why $\varphi(a, t)$ should not vary over time. The variations must be such, however, that formula (V.2) is at all times verified. This means that the variations in the fertility function at a certain age must be exactly compensated by the variations at other ages. This condition is of purely theoretical significance, however, since such compensation never takes place in reality. In an actual development with constant age distribution,[2] formula (V.2) involves invariability of the fertility function over time.

C. Semi-Malthusian populations satisfying a given condition

The fundamental property of semi-Malthusian populations makes it very simple to study the properties of processes of demographic evolution with constant age distribution associated with another condition. We shall now revert, one by one, to the examples given in chapter IV.

First example

We consider the two conditions:

(i) Constant age distribution;

(ii) Constant rate of natural variation.

In such conditions, r is no longer dependent on time and formula (V.1) shows that the survivorship function is also no longer dependent on time. In such a development, the population is Malthusian from the very beginning.

We have, however, the formula:

$$q(a) = -r - \frac{C'(a)}{C(a)}$$

[1] *The Future Growth of World Population* (United Nations publication, Sales No.: 58.XIII.2).

[2] In fact, the age distribution is almost invariable in actual developments.

and we must therefore have:

$$-r - \frac{C'(a)}{C(a)} > 0$$

or, alternatively:

$$r < -\frac{C'(a)}{C(a)} \tag{V.3}$$

We cannot therefore select at the outset an arbitrary value for the rate of natural variation which is assumed to be constant. Formula (V.3) must be complied with.

Second example

Here we consider the two conditions:

(i) Constant age distribution;

(ii) Survivorship function constant at a given age a_0. For this age a_0, formula (V.1) is written:

$$C(a_0) = b_0 e^{-r(t)a_0} p(a_0) \tag{V.4}$$

which shows that the rate of natural increase is not independent of time. This therefore brings us back to the first example. The value of r taken from formula (V.4) must of course satisfy (V.3).

Third example

Here we consider the two conditions:

(i) Constant age distribution;

(ii) Constant crude death rate.

Assuming the crude death rate as given is tantamount to assuming the rate of natural increase as given, since the crude birth rate $b = C(0)$ is already known. We therefore have $r = C(0) - d$, and if we know r this brings us back to the first example. The value of r thus determined must also satisfy formula (V.3).

Fourth example

Here we consider the two conditions:

(i) Constant age distribution;

(ii) Constant age distribution of deaths for a given age a_0.

For age a_0 we can write:

$$r(t) = \frac{b_0 d(a_0) + C'(a_0)}{-C(a_0) + d(a_0)} \quad \text{(formula II.12)}$$

which shows that $r(t)$ is not time-dependent, and thus we are once more brought back to the first example. As in the previous cases, r must, of course, satisfy formula (V.3).

Examples using actual age structure

In the numerical applications in previous chapters we have sometimes used actually observed age distributions.

These examples take on a new aspect when interpreted in the light of the properties of semi-Malthusian populations.

Let us revert to sub-set $F(r)$ corresponding to the age distribution adjusted to fit the female population of Brazil, as recorded in the censuses of 1900, 1940 and 1950. It may be recalled that we proposed, in chapter IV, to determine the population of this sub-set corresponding to a given value r_0 of the rate of natural variation. We showed that the survivorship curve of such a population is given by the formula:

$$p(a) = \frac{C(a)}{C(0)} e^{r_0 a}$$

where r_0 must satisfy the condition:

$$r_0 < -\frac{C'(a)}{C(a)}$$

By varying r_0, we obtain all the survivorship functions of the sub-set $F(r)$. It must be made clear that, in doing so we do not pose the question whether or not the population of Brazil formed part of the sub-set $F(r)$. We considered the totality of the Malthusian populations having the same age distribution as the age distribution of the population of Brazil adjusted to fit the three censuses of 1900, 1940 and 1950. This is what we termed the sub-set $F(r)$ corresponding to that age distribution. Before the population of Brazil could be included among these Malthusian populations, it would have been necessary to establish that it was a Malthusian population. We did not pose this question, and consequently it would have been imprudent to conclude that the mortality in Brazil was among the mortalities of the sub-set $F(r)$ obtained by varying r_0 within the limits indicated above. Indeed, such a conclusion was not given in chapter IV.

We now know, however, that it would have been sufficient to establish that the population of Brazil was "semi-Malthusian" in order to be in a position to give such a conclusion. According to table IV.7, there was little variation in the age distribution of the female population of Brazil between 1900 and 1950. Over that period, therefore, the Brazilian population was semi-Malthusian. In the circumstances, then, the mortalities in the sub-set $F(r)$ did in fact define the universe of possible variations in the mortality in Brazil between 1900 and 1960.

Determination of the universe to which the mortality belongs does not, of course, fully determine the mortality itself, since we have a choice between all the mortalities belonging to that universe. Often, however, we have at our disposal additional information which enables us to make the choice. In the present case, for example, we can assume that the mean annual rate of variation calculated from the census results of 1940 and 1950 gives an estimate of the annual rate of natural variation in the middle of the period, i.e., at the beginning of 1945. We thus find a rate $r = 0.0239$. The survivorship function in Brazil in 1945 is then determined by the formula:

$$p_f(a) = \frac{C_f(a)}{C_f(0)} e^{0{,}0239a}$$

The same reasoning holds good in the case of Mexico. Table IV.10 shows that from 1940 to 1950 the population of Mexico was semi-Malthusian. From 1940 to 1950, the mean rate of annual variation was equal to $r = 0.0269$. The survivorship function in Mexico in 1945 was thus equal to:

$$p_f(a) = \frac{C_f(a)}{C_f(0)} e^{0{,}0269a}$$

It must be borne in mind, in these two examples, that all the difficulties of computation resulting from the fact that the age distributions are known in discontinuous terms continue to exist.

Let us now turn to the example in chapter IV, where we seek to determine a Malthusian population *almost compatible* with the age distribution of the population of Mexico and the age distribution of deaths recorded in 1950.

In this case also, the result obtained was as follows: there is a Malthusian population whose age distribution is almost identical with that of the female population of Mexico adjusted to fit the 1930, 1940 and 1950 census results and which has an age distribution of deaths almost identical with the age distribution of deaths recorded in Mexico in 1950. This Malthusian population has a rate of natural variation $r = 0.024$ and a crude death rate $d = 0.020$.

We had, however, no grounds for concluding that those were the values for the rate of natural variation and the crude death rate in Mexico in 1950. In order that such a conclusion be true, we should show that the population of Mexico is Malthusian. We now know that it is enough to show that it is semi-Malthusian, and this is indeed the case, according to table IV.10 in chapter IV.

Since most developing countries have semi-Malthusian populations, the potential usefulness of the methods studied in the foregoing chapters is immediately apparent.

D. Reconsideration of the definition of Malthusian populations

It will be recalled that, at the beginning of this work, a Malthusian population was defined as a population in which the age distribution of the population and the mortality remained constant. It was deduced from this that many other demographic characteristics must also remain constant in such a population, particularly the age distribution of deaths. We were thus led to consider three functions of age:

The age distribution of the population $C(a)$;

The survivorship function $p(a)$;

The age distribution of deaths $d(a)$.

If we combine these three functions two by two, we obtain the following three pairs:

(1) $p(a)$, $C(a)$

(2) $p(a)$, $d(a)$

(3) $C(a)$, $d(a)$

The definition given for a Malthusian population corresponds to constancy of the first pair. Let us now consider what corresponds to constancy of the other pairs.

If $p(a)$ and $d(a)$ are invariable, we can write at time t:

$$C(a,t) = \frac{\dfrac{d(a)}{q(a)}}{\displaystyle\int_0^{\infty} \frac{d(a)}{q(a)}\,da} \qquad \text{(II.7)}$$

This is valid for all populations. This formula shows that, if $p(a)$ and $d(a)$ are invariable, $C(a)$ is invariable too. The population is therefore Malthusian.

If $C(a)$ and $d(a)$ are invariable, we are dealing with a semi-Malthusian population, and we have seen that there exist at all times between the demographic characteristics of such a population the same relationships as in a Malthusian population. We can therefore write:

$$r = \frac{C(0)d(a) + C'(a)}{-\,C(a) + d(a)}$$

This means that r is also invariable, and as:

$$p(a) = \frac{C(a)}{C(0)}\,e^{ra}$$

the survivorship function is also invariable. The population is therefore not merely semi-Malthusian, but Malthusian.

To sum up, if in a given population two of the three functions $p(a)$, $C(a)$ and $d(a)$ are invariable, that population is Malthusian. We can therefore broaden the definition of a Malthusian population given at the beginning of chapter I by stating that:

A Malthusian population is a population in which two of the three functions $p(a)$, $C(a)$ *and* $d(a)$ *are invariable.*

Let it be quite clear that only the *invariability* of the functions is required. *It is not assumed that the functions are known.*

If we assume that $p(a)$ is known, we can define particular sub-sets $H(r)$; similarly, if we assume that $C(a)$ is known, we can define a series of other particular sub-sets $F(r)$. Thus, there emerges the possibility of defining a third series of particular sub-sets if we assume $d(a)$ to be known. We shall term this third series $G(r)$, and it will be studied in chapter VI below. One last remark: the three series $H(r)$ and $G(r)$ are not really distinct from each other. When the functions $p(a)$, $C(a)$ and $d(a)$ vary as well as r, we obtain all possible Malthusian populations. The three series differ only in the classification of all these possible Malthusian populations.