Chapter III

MALTHUSIAN POPULATIONS WITH KNOWN MORTALITY FUNCTIONS, CONSIDERED AS THE LIMIT IN A DEMOGRAPHIC EVOLUTION

A. Introduction

Throughout chapter II, the populations considered were Malthusian populations, i.e., populations with constant mortality and age distribution, and in addition the mortality was assumed to be known for each sub-set H(r). Finally, knowledge of one additional condition made it possible to identify one population, or at least a small finite number of populations in each sub-set H(r).

Let us take, for example, the case of a stable population corresponding to a constant and known fertility function while belonging to a sub-set H(r). This stable population has been defined by the following properties:

(a) *Constant* age distribution;

(b) *Constant* and *known* mortality;

(c) *Constant* and *known* fertility.

In another example (the first example in chapter II) we arrived at a population with the following properties:

(a) *Constant* age distribution;

(b) *Constant* and *known* mortality;

(c) *Constant* and *known* rate of increase.

We can easily spell out the conditions corresponding to the various cases envisaged in chapter II, and in each case we shall find a constant age distribution associated with a known and constant mortality, plus one other condition.

Let us imagine that in each of these cases we dispense with the first condition, i.e., the constant age distribution. We are then left with a set of demographic conditions which can be imposed on a population from a given arbitrary initial state. We thus set in motion a process of demographic evolution, but is it possible to bring out the simple laws of such a process? That is the problem which we propose to examine next.

It may be recalled that we have already dealt experimentally, in chapter I, with the case where the given demographic conditions are those of constant mortality and constant fertility. We have shown in several actual cases that in a process of demographic evolution under these conditions the population approaches the stable Malthusian population corresponding to the given laws of mortality and fertility. This is the case dealt with by Lotka in the work already referred to.[1]

Various problems involved

We can obviously imagine a wide range of problems of the type which we have defined above and of which the concept of a limit stable population is an example. All the problems in this range are not equally important, however, and they can be classified in various categories:

(a) *Determinate problems.* These are problems leading to population evolution which is not impossible. The way to find out whether a problem is determinate or not is to try to compute a population projection, as was done in chapter I with Lotka's theorem. This theorem is the very model of the determinate problem. Whatever the functions $p_f(a)$ and $\varphi_f(a)$ and whatever the initial age structure may be, it is always possible to compute a projection. Other cases can be envisaged, however, where the problem is determinate only *in general terms.* Let us suppose, for example, that we assume the age structure $C_f(a)$ and the rate of increase r to be independent of time. The projection will then be calculated as follows: beginning with a female population $N_f(t)$ having an age distribution $C_f(a)$ at time t, we calculate the population at time $t+1$ by the formula: $N_f(t+1)=(1+r)N_f(t)$ and we distribute $N_f(t+1)$ according to the distribution $C_f(a)$. We thus obtain the age distribution of the population at time $t+1$. By repeating the same operation, we can therefore compute a projection of the age distribution of the population. If, however, we wish to follow persons of an initial age group throughout the projection computed in this way, we must find a number of persons which will consistently decrease, for if r is too great we may very well see the number increasing rather than decreasing. The problem which we have posed is, therefore, not always determinate. It may, in some cases, become impossible.

We see at the same time what is meant by "impossible". What we are referring to is a logical "impossibility"—in the case in question, a survivorship function which does not decrease. Provided that there is no logical impossibility, however, we classify a case as "possible", even if this logical possibility leads to a type of population development which is very unlikely to occur.

(b) *Indeterminate problems.* Let us suppose that we assume a crude birth rate b and a crude death rate d. Starting from an initial population, we can obviously compute the total number of persons in the population at any time in the future, but there are an infinite number of mortality and fertility functions through which such results can be attained, and we can hardly make any definite statement concerning these functions. Thus, the problem can be considered indeterminate.

(c) *Indeterminate problems which become determinate when it is also assumed that the mortality and fertility functions form part of the "universe" of model mortalities and fertilities.* When this assumption is made, the preceding problem, for example, becomes determinate in most cases. For a given age structure there is, generally speaking, only one model mortality function giving a crude death rate equal to d and only one model fertility

[1] Alfred J. Lotka, *Théorie analytique des associations biologiques ; Deuxième partie* (Paris, Hermann, 1939), 149 pp.

function giving a crude birth rate equal to b. The reservation "generally speaking" is necessary because it may happen, depending on the given age structure, that either there is no model mortality or model fertility compatible with the values b and d or there are several such functions.

(d) *Impossible problems.* We cannot, for example, assume as given the mortality and fertility functions as well as the rate of variation. The first two functions are sufficient to determine the problem; the third is superflous and may make the problem impossible.

Whe shall proceed to consider only problems of the first type, i.e., determinate problems. It will therefore be necessary to verify, in each case, that the conditions which we set are not contradictory or indeterminate at any time during the evolutionary process.

These "determinate" problems correspond to the various examples of the determination of Malthusian populations studied in chapter II. The examples are the same, in fact, except that the condition of an invariable age structure has been dispensed with. We shall take them up successively, after first defining the general conditions of possibility.

Conditions of possibility

Starting from a given initial population, we assume that the mortality of both sexes remains constant and that one other demographic characteristic also remains constant. In order to see whether the process of demographic evolution defined in this way leads to impossibilities or not, we need only show that it is in fact possible in such circumstances to compute, on the basis of the initial state (period zero), the population at the next period (period 1), then at period 2, 3 and so forth, or, in other words, to show that a "population projection" can be computed. For an understanding of how certain contradictions may arise, it may be best to take a specific example.

Let us take as our initial state (state zero) the female population of Eastern Germany according to the 1957 census, which we have already used in chapter I. This population is given in five-year age groups in the second column of table III.1. Starting from this initial state, let us keep the mortality constant at level 80 of the intermediate model life table (expectation of life at birth for both sexes of 60.4 years). The survival ratios from one age group to the next over a five-year period are given in the last column of table III.1. By multiplying each term of the second column by the corresponding survival ratio, we obtain the number of survivors five years later. We can thus determine for each group of five years the female population aged 5 and over. We thus find that: $N_{5\ \&\ \text{over}} = 8{,}351{,}319$ for an initial population of $N_{0\ \&\ \text{over}} = 9{,}031{,}093$ (these two figures are given in the last line of table III.1).

Let us represent by B the average annual female births during the five years under consideration, and let b, d and r represent the average annual female rates of birth, death and natural increase, respectively, during those five years.

The survival ratio to the year 5 of mean annual births B is 0,9208 (the first figure in the last column of table III.2). In other words, girls aged 0-4 will number, in year 5:

$$0.9208\ \text{B} \times 5 = 4.604\ \text{B}$$

Table III.1. Five-year projections based on the female population of Eastern Germany according to the 1957 census

Age group (years)	Female population in year 0 (a)	Female population in year 5	Survival ratio (c) from one age group to the next 0.9208 (d)
0-4	440 981		0.9731
5-9	706 869	429 119	0.9914
10-14	410 485	700 790	0.9906
15-19	617 750	406 626	0.9865
20-24	748 500	609 410	0.9839
25-29	536 107	736 449	0.9826
30-34	543 514	526 779	0.9811
35-39	545 370	533 243	0.9783
40-44	490 705	533 535	0.9727
45-49	715 494	477 309	0.9631
50-54	712 084	689 092	0.9482
55-59	680 800	675 198	0.9238
60-64	595 017	628 923	0.8825
65-69	495 159	525 103	0.8162
70-74	372 892	404 149	0.7197
75-79	240 681	268 370	0.5955
80-84	127 718	143 326	0.3576 (e)
85 and over	50 967	63 898	
All ages	9 031 093	8 351 319 (b)	

(a) Female population according to 1957 census.
(b) Total population aged 5 and over.
(c) Survival ratios corresponding to the level-80 model life table (expectation of life at birth for both sexes of 60.4 years).
(d) Survival ratio at the end of the fifth year from births spread evenly over the five years.
(e) Survival ratio over the five-year period of persons aged 80 and over.

while the total population in year 5 will be:

$$N_{5\ \&\ \text{over}} + 4.604\ \text{B} = 8{,}351{,}319 + 4.604\ \text{B}$$

The mean population during the five years is:

$$\frac{9{,}031{,}093 + 8{,}351{,}319 + 4.604\ \text{B}}{2} = 8{,}691{,}206 + 2.252\ \text{B}$$

and the mean annual increase in population is:

$$\frac{(8{,}351{,}319 + 4.604\ \text{B} - 9{,}031{,}093)}{5} =$$
$$= -\ 135{,}954.8 + 0.9208\ \text{B}.$$

We then have the following formulae for the rates b, d and r:

$$b = \frac{\text{B}}{8{,}691{,}206 + 2.252\ \text{B}}$$

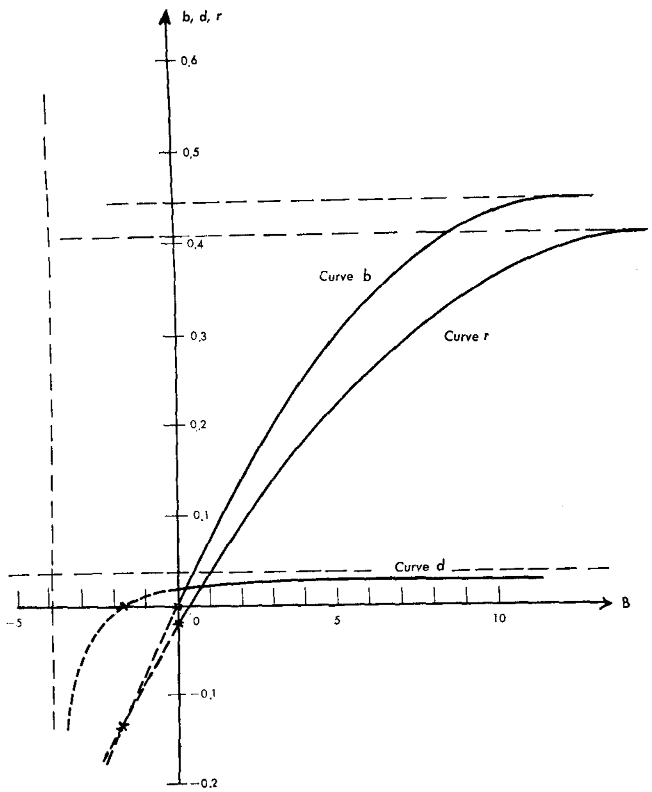
$$d = \frac{135{,}954.8 + 0.0792\ \text{B}}{8{,}691{,}206 + 2.252\ \text{B}}$$

$$r = \frac{-135{,}954.8 + 0.9208\ \text{B}}{8{,}691{,}206 + 2.252\ \text{B}}$$

Let us examine the variations of b, d and r as a function of B. These quantities are homographic functions of B, and thus the corresponding curves are branches of equilateral hyperbolae. In graph III.1, the horizontal axis represents B and the vertical axis represents b, d, and r.

The three curves have in common as asymptote a straight line whose abscissa is given by:

$$\text{B} = -\ \frac{8{,}691{,}206}{2.252} = -\ 3{,}859{,}000$$



Graph III.1. Determination of the conditions of compatibility of system III.A in the text

Each of the curves also has as a horizontal asymptote whose ordinate is given by a straight line:

$$b_a = 0.440$$
$$d_a = 0.0352$$
$$r_a = 0.4088$$

We also, of course, have: $b_a - d_a = r_a$.

Finally, the smallest value that B can have is B = 0, and for this value of B we have:

$$b_0 = 0$$
$$d_0 = 0.0156$$
$$r_0 = -0.0156$$

In addition, of course, we have: $b_0 - d_0 = r_0$.

Finally, b, d, and r can vary over that part of their curves which is shown as a solid line in graph III.1. We therefore see that these three quantities cannot assume arbitrary values during the five years under consideration. We must have:

$$\left.\begin{array}{r} 0 < b < 0.4440 \\ 0.0156 < d < 0.0352 \\ -0.0156 < r < 0.4088 \end{array}\right\} \qquad \text{(III.A)}$$

If the conditions defining the process of demographic evolution involve values of b, d *or* r *which do not satisfy system (III.A), then it will not be possible to compute the projection.*[2]

[2] The fact that human fertility rates cannot exceed certain values causes further limitation as to the possible variation of *b*, *d* and *r*. This is a different problem, however. We are concerned only with *logical* contradictions.

It should be noted, in passing, that knowledge of the age-specific female fertility rates determines the mean annual number of births without ever introducing any contradiction. In order to obtain B, all we have to do is to apply the fertility rates to the 15-49 age groups at year zero and year 5. The average of the results obtained is the number B which is sought.

Let us suppose that we have taken demographic characteristics satisfying the conditions under III.A. We are thus certain of being able to calculate the population at year 5 on the basis of the initial state. If we wish to continue beyond year 5 and calculate the population in year 10, however, we are confronted with a problem similar to that described above, and we shall, in fact, encounter such a problem at each new five-year period. The horizontal asymptotes of the curves representing the variation of b, d and r are always the same at each period, because their ordinates depend only on the mortality, but the vertical asymptote common to the three curves changes at each period, although its abscissa is always negative. The ordinates at the origin of the curves representing d and r also change at each period. Hoewever, we always have: $d_0 = -r_0$, and d_0 is always positive.

Finally, if d_m represents the highest value of d_0 in the process and d_m the lowest value of d_0, then b and r must satisfy the conditions:

$$\left.\begin{array}{r} 0 < b < 0.4440 \\ 0 < d_m < d < 0.0352 \\ -d_m < 0 < r < 0.4088 \end{array}\right\} \qquad \text{(III.A}bis\text{)}$$

In all that follows we shall assume that we are dealing with a case where these conditions are satisfied.

B. Limit stable population

Let us consider a given initial population in which female mortality and female fertility remain invariable. We have seen above that, in such a process, the conditions under III.A*bis* are always satisfied.

Let us consider only the female population. The females of age a at time t, who are the survivors of the $B(t-a)$ girls born at time $t-a$, number $B_f(t-a)p_f(a)$. Consequently, the total female births at time t are written:

$$B_f(t) = \int_0^\omega B_f(t-a)p_f(a)\varphi_f(a)da \qquad \text{(III.1)}$$

This is the basic equation of the process of demographic evolution with constant mortality and fertility.

In order to solve this equation, we seek a solution of the form:

$$B_f(t) = A_1e^{r_1t} + A_2e^{r_2t} + A_3e^{r_3t} + \ldots \qquad \text{(III.2)}$$

We can thus write:

$$A_1e^{r_1t} + A_2e^{r_2t} + A_3e^{r_3t} + \ldots$$
$$= A_1e^{r_1t}\int_0^\omega e^{-r_1a}p_f(a)\varphi_f(a)da$$
$$+ A_2e^{r_2t}\int_0^\omega e^{-r_2a}p_f(a)\varphi_f(a)da + \ldots$$

We see that if r_1, r_2, r_3 etc., are the roots of the equation:

$$\int_0^\infty e^{-ra} p_f(a)\varphi_f(a)da = 1 \qquad \text{(III.3)}$$

formula III.2 is the solution of III.1.

We recognize in equation III.3 formula II.14 linking the rate of natural increase, the mortality and the fertility in a stable population. We have seen that this equation has a single real root ρ, which we termed the intrinsic rate of natural variation.

This root is associated with an infinity of complex roots, which possess the following two properties:

(1) Let $x + iy$ be one of these roots. We then have:

$$\int_0^\infty e^{-(x+iy)a} p_f(a)\varphi_f(a)da = 1$$

which is written:

$$\int_0^\infty e^{-xa} [\cos y\, a - i \sin y\, a]\, p_f(a)\varphi_f(a)da = 1$$

Where "i" represents $\sqrt{-1}$.

In separating the real part from the imaginary part, we obtain:

$$S \begin{cases} \int_0^\infty e^{-xa} \cos y a p_f(a)\varphi_f(a)da = 1 \\ \int_0^\infty e^{-xa} \sin y a p_f(a)\varphi_f(a)da = 0 \end{cases}$$

For a given value of y, we have:

$$1 = \int_0^\infty e^{-xa} \cos y a p_f(a)\varphi_f(a)da < \int_0^\infty e^{-xa} p_f(a)\varphi_f(a)da$$

and as:

$$\int_0^\infty e^{-\varrho a} p_f(a)\varphi_f(a)da = 1$$

we therefore have:

$$\int_0^\infty e^{-\varrho a} p_f(a)\varphi_f(a)da < \int_0^\infty e^{-xa} p_f(a)\varphi_f(a)da$$

from which it follows that:

$$\varrho > x.$$

The intrinsic rate ρ is therefore superior to all the quantities x.

(2) If $(x + iy)$ is a root, then the conjugate complex number $(x - iy)$ is also a root, since system (S)is satisfied by both quantities. As the number of births $B_f(t)$ is necessarily a real number, the coefficients A_n corresponding to the two conjugate roots are equal, and we finally have for $B_f(t)$ the formula:

$$B_f(t) = A_1 e^{\rho t} + A_2 e^{x_2 t} \cos y_2 t + A_3 e^{x_3 t} \cos y_3 t + \dots$$

in which ρ is higher than x_2, x_3 etc.

We see that the imaginary roots introduce oscillations in the number of births. Since ρ is greater than all the quantities x, as the time t increases indefinitely the term $A_1 e^{\varrho t}$ becomes preponderant over all the others, the latters giving rise to damped oscillations.

In other words, the number of births at time t asymptotically approaches:

$$B_f(t) \rightarrow A_1 e^{\varrho t}$$

i.e., it tends to follow the law of births in the stable population corresponding to the laws $p_f(a)$ and $\varphi_f(a)$. The population itself approaches this stable state. It is this result, arrived at empirically in chapter I, which constitutes what may be termed Lotka's theorem.

We showed in the first part of this chapter how to calculate the intrinsic rate, the death rate and the age distribution of the stable state. Generally speaking, we confine ourselves to a consideration of these characteristics. For completeness, however, it is also necessary to calculate the constants A_1, A_2, A_3 ... Annex I gives details of a method which makes it possible to determine the constant A_1 corresponding to the intrinsic rate ρ and thus permits the complete calculation of the stable state.

$$B_f(t) = A_1 e^{\varrho t} \qquad \text{(III.4)}$$

The formula at which we arrive is the following:

$$A_1 = \int_0^v \frac{K_f(a, 0)}{p_f(a)} G(a) e^{\varrho a} da = \int_0^v S(a)da$$

Where

$$S(a) = \frac{K_f(a, 0)\, G(a)}{p_f(a)}$$

$K_f(a, o)$ represents the number of women of age a at the starting point and $G(a)$ is a function of a which does not depend on the initial conditions and is practically the same, whatever the human fertility and mortality functions may be.[3]

In the stable state, the total population $N(t)$ is obtained by dividing the births by the birth rate:

$$b = \frac{1}{\int_0^\infty e^{-\varrho a} p_f(a)da}$$

Thus we have:

$$N(t) = e^{\varrho t} \int_0^\infty e^{-\varrho a} p_f(a)da \int_0^\infty S(a)da \qquad \text{(III.5)}$$

Reconsideration of the special case of chapter II

We pointed out earlier that the computation of the intrinsic rate ρ becomes very easy when the fertility function $\varphi_f(a)$ is reduced to a single value $\varphi_f(27.5)$. It was stated that in such circumstances we were dealing with the "special case". We shall see that in this case the oscillations in the number of births corresponding to the complex roots of the fundamental equation do not disappear with the passage of time.

The basic equation is written:

$$e^{-27.5r} R_0 = 1$$

If we assume that $r = x + iy$, the equation becomes:

$$[\cos 27.5y + i \sin 27.5y] e^{-27.5x} = \frac{1}{R_0}$$

[3] The values of $G(a)$ are to be found in table III.2 below.

which breaks down into two equations:

$$\begin{cases} e^{-27.5x} \cos 27.5y = \dfrac{1}{R_0} \\ e^{-27.5x} \sin 27.5y = 0 \end{cases}$$

whence we have:

$$\sin 27.5y = 0$$

$$e^{-27.5x} = \frac{1}{R_0}$$

or, finally: [4]

$$27.5y = 2\lambda\pi$$

$$x = \varrho$$

The number of births at time t is therefore:

$$B_f(t) = A_1 e^{\varrho t} + A_2 e^{\varrho t} \frac{\cos 2\pi t}{27.5} + A_3 e^{\varrho t} \frac{\cos 6\pi t}{27.5} + \ldots$$

or:

$$B_f(t) = e^{\varrho t} A_1 + \Sigma A_n \cos \frac{2nt}{27.5}$$

Thus, births oscillate continuously a round the exponential term:

$$A_1 e^{\varrho t}$$

There will therefore not be any damping out of the oscillations. In this case, the stable state no longer appears as a limit state, but rather as a mean value. It should be noted that *the special case is the only case* where the oscillations due to the complex roots of the fundamental equation are not damped out.

Numerical application

Tables III.2 and III.3 give an example of the application of the formula III.4 (for determining the absolute number of births, $B_f(t)$ in the process of projection making based on the population of Thailand in 1955.

The annual number of births in the stationary population is:

$$B_f(t) = 275{,}634$$

The stationary population is obtained by dividing the births by the crude female birth rate, which is the same thing as multiplying the births by the female expectation of life at birth (see table II.11): $275{,}634 \times 62.05 = 17{,}103{,}000$.

In the second stable population the annual number of female births at time $t = 0$ is $B_f(^0) = 308{,}380$.

If we divide this number by the crude female birth rate $b_f = 0.02179$ (see table II.11), we obtain for the total number of persons in the population at time $t = 0$:

$$N_f(0) = \frac{B_f(0)}{b_f} = \frac{308{,}380 \times 1{,}000}{21.79} = 14{,}152{,}000$$

In the case of the third stable population calculated in chapter I on the basis of the population of Thailand and the three stable populations calculated in the same chapter on the basis of the population of Eastern Germany in 1957, we shall confine ourselves to giving the results of the computations (table III.4). It was with the aid of these results that we plotted the straight lines on graphs I.2, I.4, I.9, I.10 and I.11 in chapter I.

C. A limit Malthusian population with constant mortality and a constant and given crude birth rate

In this second problem, we shall suppose that, starting off with a given initial condition, we keep the mortality $p_f(a)$ and the crude birth rate b_f constant. This case corresponds to the third example in chapter II.

We assume, of course, that the condition imposed on b_f in the system of inequalities III.A *bis* above is satisfied; in fact, the relevant in equality is almost always satisfied for b_f, since the uppser limit of b_f is always much higher than the crude birth rates encountered in the human species.

[1] The solution $27.5k = \lambda\pi$ would give a negative value for $\cos 27.5y$, which is impossible, since we have:

$$\cos 27.5y = \frac{e^{27.5x}}{R_0}$$

TABLE III.2. COMPUTATION OF THE ANNUAL NUMBER OF FEMALE BIRTHS IN THE STATIONARY POPULATION ON THE BASIS OF THE POPULATION OF THAILAND IN 1955

Median age a	*Age group (years)*	*Initial number of women* $K_a(0)$	*Stationary female population* (a) L(b) *(per 100 000 births)*	*Ratio of the two preceding columns*	$G(a)$ (b)	*Product of the two preceding columns (stationary births per 100 000)*	*Product of preceding column and median age* (c)
2.5	0-4	1 809 000	460 386	3.930	0.18	0.70740	1.7685
7.5	5-9	1 397 000	448 010	3.118	0.18	0.56124	4.2093
12.5	10-14	1 196 000	444 150	2.692	0.18	0.48456	6.0570
17.5	15-19	1 110 000	439 970	2.522	0.17	0.42874	7.5030
22.5	20-24	987 000	434 040	2.274	0.13	0.29562	6.0515
27.5	25-29	824 000	427 035	1.930	0.09	0.17370	4.7768
32.5	30-34	658 000	419 610	1.568	0.05	0.07840	2.5480
37.5	35-39	549 000	411 672	1.334	0.02	0.02668	1.0005
	ALL AGES					2.75634	34.5146

(a) Intermediate model life table corresponding to an expectation of life at birth for both sexes of 60.4 years.
(b) The function $G(a)$ is the same for all populations. See annex I for more details.
(c) The computation shown in this last colum is used in annex I.

Table III.3. Computation of the annual number of female births in the stable population on the basis of the population of Thailand in 1955 [a]

Median age a	Age group (years) [a]	$e^{\rho a}$	Stationary births [b] (per 100 000)	Product of the two preceding columns (initial [c] stable births)	Product of the preceding column and the "median age" column [d]
2.5	0-4	1.0220	0.70740	0.7230	1.8075
7.5	5-9	1.0674	0.56124	0.5990	4.4925
12.5	10-14	1.1159	0.48456	0.5408	6.7600
17.5	15-19	1.1654	0.42874	0.4996	8.7430
22.5	20-24	1.2173	0.29562	0.3598	8.0955
27.5	25-29	1.2702	0.17370	0.2206	6.0665
32.5	30-34	1.3268	0.07840	0.1040	3.3800
37.5	35-39	1.3857	0.02668	0.0370	1.3875
	All ages			3.0838	40.7325

[a] Intermediate fertility model with constant distribution corresponding to a gross reproduction rate of 1.50 and model life table corresponding to an expectation of life at birth for both sexes of 60.4 years.
[b] Figures taken from table II.11.
[c] Figures related to time $t = 0$.
[d] The computation shown in this last column is used in annex I.

Table III.4. Absolute annual number of female births and absolute number of women in three stable populations computed on the basis of (*a*) population of Eastern Germany in 1957; (*b*) population of Thailand in 1955

Gross reproduction rate	Births $B_f(t)$	Population (millions) $N_f(t)$
(a) *Based on East Germany, 1957*		
1.50	$148\,980\, e^{0.0087t}$	$6\,802\, e^{0.0087t}$
1.17	128 980	8 003
0.75	$102\,060\, e^{-0.0157t}$	$12\,064\, e^{-0.0157t}$
(b) *Based on Thailand, 1955*		
1.50	$308\,380\, e^{0.0087t}$	$14\,152\, e^{0.0087t}$
1.17	275 634	17 103
0.75	$228\,530\, e^{-0.0157t}$	$27\,013\, e^{-0.0157t}$

At time t, we can write for the total female population $N_f(t)$:

$$N_f(t) = \int_0^\omega B_f(t-a)p_f(a)da$$

and the births at time t are written:

$$B_f(t) = b_f \int_0^\omega B_f(t-a)p_f(a)da$$

Thus, we have an equation of the same type as equation III.3, and the solution sought is:

$$B_f(t) = A_1e^{\rho t} + A_2e^{r_2t} + A_3e^{r_3t} + \dots$$

where ρ is the real root and r_2, r_3 ... are the complex roots of the equation:

$$b_f\int_0^\omega e^{-ra}p_f(a)da = 1$$

or

$$\int_0^\omega e^{-ra}p_f(a)da = \frac{1}{b}$$

The population approaches the Malthusian population corresponding to the function $p_f(a)$ and the crude birth rate b_f.

D. A limit Malthusian population with constant mortality and constant rate of natural variation

Let us now suppose that, starting off with a given initial state, we keep the mortality $p_f(a)$ and the rate of natural variation r_0 constant. We assume, of course, that the value r_0 selected for the rate of variation satisfies the relevent inequality in system III.A *bis*. It should be noted that, if r_0 is positive, this inequality is always satisfied, since the upper limit of r in system III.A *bis* is always much higher than the values encountered in the human species.

We can write for the female population:

$$N_f(t) = N_f(0)e^{r_0t}$$

However, we also have:

$$N_f(t) = \int_0^\omega B_f(t-a)p_f(a)da$$

whence we have the equation:

$$N_f(0)e^{r_0t} = \int_0^\omega B_f(t-a)p_f(a)da$$

If we seek a solution of the form:

$$B_f(t) = A_1e^{r_1t} + A_2e^{r_2t} + A_3e^{r_3t} + \dots$$

we have :

$$N_f(0)e^{r_0t} = A_1e^{r_1t}\int_0^\omega e^{-r_1a}p_f(a)da + A_2e^{r_2t}\int_0^\omega e^{r_2a}p_f(a)da + \dots$$

If we assume that:

$$r_1 = r_0$$

and

$$A_1 \int_0^\omega e^{-r_0 a} p_f(a) da = N(0)$$

and if r_2, r_3, ... are the complex roots of the equation:

$$\int_0^\omega e^{-ra} p_f(a) da = 0 \qquad \text{(III.6)}$$

then the expression:

$$B_f(t) = \frac{N(0)}{\int_0^\omega e^{-r_0 a} \; p_f(a) da} e^{r_0 t} + A_2 e^{r_2 t} + A_3 e^{r_3 t} + \ldots$$

is the solution sought.

The crude birth rate is written:

$$b_f(t) = \frac{1}{\int_0^\omega e^{-r_0 a} p_f(a) da} + A_2 \frac{e^{(r_2 - r_0)t}}{N(0)} + A_3 \frac{e^{(r_3 - r_0)t}}{N(0)} + \ldots$$

There is an upper limit to the crude birth rate, so that we have in reality:

$$b_f = r_0 - d_f, \text{ and thus } b_f < r_0$$

It must therefore be true that: $r_2 - r_0 < 0, r_3 - r_0 < 0$, and so on. r_0 is therefore greater than all the r_2, r_3, r_4 and so on, and the population tends towards the Malthusian population of the set $H_0(r)$ connected with $p_f(a)$, whose rate of natural variation is equal to r_0.

E. A limit Malthusian population with constant mortality and constant crude death rate

Let us now suppose that, starting from any initial state, we keep the mortality $p_f(a)$ and the crude death d_0 constant. This case corresponds to the fourth example in chapter II.

Here again, we suppose that the value d_0 chosen for the crude death rate satisfies the corresponding inequality of system III. A *bis*. We have already seen that the inequality corresponding to b_f in system III.A *bis* is practically always satisfied and that the inequality corresponding to r is very frequently satisfied (in practice, it is sufficient that r should be positive in order for the inequality to be satisfied). The same does not apply in the case of d, however.

In the case given as an example in table III.1 (where the initial population is the female population of Eastern Germany according to the census of 1957 and the mortality is that of level-80 of the intermediate model life table), d had to be under 35.2 per thousand and over 15.6 per thousand, although the latter limit, which was calculated only on the basis of the first years of the projection, might prove inadequate in the computations. Thus, the problem posed is impossible unless d_0 falls within quite narrow limits.

The study of the conditions governing the possibility or impossibility of the problem is thus of the greatest importance here, and it can be carried out only when the initial state, the mortality and the crude death rate are known. In succeeding pages it will be assumed that the compatibility of the conditions imposed has been verified.

The number of females aged a at time t is:

$$K_f(a,t) = B_f(t-a) \, p_f(a)$$

The number of female deaths at time t is:

$$\int_0^\omega B_f(t-a) p_f(a) q_f(a) da$$

and consequently the crude death rate is:

$$d_0 = \frac{\int_0^\omega B_f(t-a) p_f(a) q_f(a) da}{\int_0^\omega B_f(t-a) p_f(a) da}$$

We thus have the equation:

$$\int_0^\omega B_f(t-a) p_f(a) \, [d_0 - q_f(a)] da = 0$$

The number of births is of the form:

$$B_f(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t} + A_3 e^{r_3 t} + \ldots$$

where r_1, r_2, r_3 etc., are the roots of the equation:

$$\int_0^\omega e^{-ra} p_f(a) [d_0 - q_f(a)] da = 0$$

Taking account of the formula:

$$p_f(a) q_f(a) = - p_f'(a)$$

and integrating by parts, we finally have the equation:

$$(d_0 + r) \int_0^\omega e^{-ra} p_f(a) da = 1 \qquad \text{(III.7)}$$

This is an r equation which can be written:

$$d_0 = \frac{1}{\int_0^\omega e^{-ra} p_f(a) da} - r = F(r)$$

We have already encountered the expression $F(r)$ in connexion with the fourth example in chapter II. Graph II.9 gave the form of the curve representing the variation of $F(r)$ as a function of r. The abscissae of the points of intersection of this curve with the straight line of the ordinate d_0 are the real roots of equation (III.7). There are zero, one or two real points of intersection, depending on the value of d_0. Equation III.7 thus has zero, one or two real roots. It also has an infinite number of complex roots conjugate in pairs. Let ρ_1 and ρ_2 be the real roots and $x_3 + iy_3$, $x_4 + iy_4$ etc., the sequence of the complex roots. We have the following formula for the births at time t:

$$B_f(t) = A_1 e^{\rho_1 t} + A_2 e^{\rho_2 t} + A_3 e^{x_3 t} \cos y_3 t + A_4 e^{x_4 t} \cos y_4 t + \ldots$$

When the time increases indefinitely, the term corresponding to the largest of the quantities ρ_1, ρ_2, x_3, x_4 etc., becomes preponderant over all the others. In the preceding examples, the term which thus became preponderant over all the others was always a real one, but this is not the case here. This is obvious when equation III.7 does not have any real roots, but it can happen even when real roots exist. If it is one of the quantities in x which is the greatest, the births are expressed in the form:

$$B = Ae^{xt} \cos yt$$

The births continually oscillate around a mean value, and the limit population is no longer a Malthusian population.

We shall confine ourselves to these few cases of limit populations, but as in the examples in chapter II it would be easy to imagine many other cases.

In the next chapter, we shall leave the sub-sets H(r) associated with a given mortality and revert to the general subject of all Malthusian populations, considering it this time as an infinity of sub-sets associated with a given age structure or, in other words, the sub-sets F(r) to which we have already referred briefly in chapter I.