Chapter II

MALTHUSIAN POPULATIONS WITH KNOWN MORTALITY FUNCTIONS (sub-sets $H(r)$)

The way in which a sub-set of Malthusian populations $H_0(r)$ could be defined for each survivorship function $p_0(a)$ was described in the preceding chapter. It was also stated that the knowledge of an additional condition made it possible to determine from such a sub-set a particular Malthusian population, or at least a limited number of Malthusian populations which satisfied that condition. In particular, knowledge of the fertility function makes it possible to single out a particular Malthusian population called "stable population".

Chapter II resumes the study of the sub-sets $H(r)$ with reference to actual cases, or, more precisely, the determination of particular populations subject to a knowledge of an additional condition.

A. Fundamental formulae

Notation used

The study of the sub-sets $H(r)$ involves most of the mathematical functions used in pure demography. The notation employed has been more or less fixed by usage, but as the available notation has been built up over the years in a somewhat unco-ordinated manner, it sometimes happens that there are too many symbols or, conversely, that different functions are expressed by almost identical symbols. We must therefore make it clear what notation is to be used in this study, and this notation is set out in table II.1. Generally speaking, each function can refer to the female population, the male population or both together. Symbols are given the subscript f when used for the female population and the subscript m when used for the male population. For example, the survivorship function at age a is written $p(a)$ for both sexes together, $p_f(a)$ for the female population, and $p_m(a)$ for the male population.

In table II.1, the only variable is the age (or age group). It frequently happens, however, that the functions under consideration depend also on other variables, such as time. In such a case, the same symbols are used, but the additional variables are written next to the age. For example, a female survivorship depending on time is written $p_f(a, t)$.

Examination of table II.1 shows that for any given concept there are generally two functions: one valid continuously (instantaneous indices) and the other valid discontinuously (age-specific indices). Generally speaking, the functional notation of the form $f(x)$ is reserved for continuous notation, while the notation using indices of the form i_m is used for discontinuous notation.

TABLE II.1. THE MAIN SYMBOLS USED TO REPRESENT THE FUNCTIONS ENCOUNTERED IN THIS STUDY

Survivorship function at age a	$p(a)$
Instantaneous death rate at age a	$q(a)$
Probability of death within n years from age a	${}_nq_a$
Death rate for a specific age group * (e.g., the age group 20-24)	m_{20-24}
Stationary population * (e.g., population within an age group 20-24)	L_{20-24}
Life expectancy at age a	e_o^a
Instantaneous probability of survival *	$s(a)$
Probability of survival from age a to age $a + n$	${}_ns_a$
Survival ratio from one age group to the next, * $\frac{L_{25-29}}{L_{20-24}}$	S_{20-24}
Instantaneous fertility rate at age a	$\varphi(a)$
Fertility rate for a specific age group * (e.g., 20-24)	f_{20-24}
Age distribution of fertility rates	$F(a)$
Distribution of fertility rates by age groups * (e.g., for the age group 20-24)	F_{20-24}
Total number of persons of all ages in the population	K
Number of persons aged between a and $a + da$ years	$K(a)$
Number of persons in a given age group * (e.g., 20-24)	K_{20-24}
Age distribution of the population at age a	$C(a)$
Distribution of the population by age groups * (e.g., 20-24)	C_{20-24}
Logarithmic derivative of $C(a)$ with changed sign:	
(1) In continuous form: $\frac{-C'(a)}{C(a)} = Q(a)$	

TABLE II.1 (*continued*)

(2) In discontinuous form:

$$-\frac{C(a+5)-C(a)}{C(a)} = {}_5Q_a$$

Number of deaths between a and $a + da$ years of age	$D(a)$
Number of deaths of a given age group * (e.g., 20-24)	$D_{20\text{-}24}$
Age distribution of deaths	$d(a)$
Distribution of deaths by age groups * (e. g., the age group 20-24 whose median age is 22.5 years)	$d_{20\text{-}24}$
Number of children born to mothers aged between a and $a + da$ years	$B(a)$
Number of children born to mothers of a given age group * (e.g., 20-24)	$B_{20\text{-}24}$
Number of births at a given point in time	$B(t)$
Crude birth rate	b
Number of deaths at a given point in time	$D(t)$
Crude death rate	d
Age at the beginning of the female reproductive period	u
Age at the end of the female reproductive period	v
Age limit of human life	ω
Masculinity at birth (number of male births per female birth)	m

---

* When the interval of the age group in question is known, only the lower age is given in the notation, e.g., m_{20}, L_{20}, S_{20}, f_{20}, F_{20}, K_{20}, C_{20}, D_{20}, d_{20}, B_{20}, instead of m_{20-24}, L_{20-24}, S_{20-24}, f_{20-24}, F_{20-24}, K_{20-24}, C_{20-24}, D_{20-24}, d_{20-24}, B_{20-24}.

DEFINITIONAL RELATIONS AMONG VARIOUS FUNCTIONS

The various functions of table II.1 are not independent. By definition, they are linked by certain relations, of which the main ones are the following:

(*a*) For mortality functions, we have the following well-known equations:

$$q(a) = -\frac{p'(a)}{p(a)} = \frac{D(a)}{K(a)}$$

$$_nq_a = \frac{p(a) - p(a+n)}{(pa)}$$

($_nq_a$ tends to $q(a)$ when n approaches zero)

$$e_0^o = \int_0^\infty p(a)da$$

$$e_0^a = \frac{1}{p(a)}\int_a^\infty p(a)da$$

$$m_{20\text{-}24} = \frac{D_{20\text{-}24}}{K_{20\text{-}24}}$$

(*b*) For fertility functions we have the following equations:

$$\varphi(a) = \frac{B(a)}{K(a)}$$

$$f_{20\text{-}24} = \frac{B_{20\text{-}24}}{K_{20\text{-}24}}$$

(*c*) The values relating to a given age group are the integrals of the corresponding instantaneous functions in that age group.

If we set aside the two extremities of life and if the interval k between the age groups is not too wide, we find that the value of a given attribute corresponding to a given age group is almost equal to k times the instantaneous value corresponding to the median age of that age group.

Thus, for five-year age groups ($k = 5$) we have:

$$L_{20\text{-}24} = \int_{20}^{25} p(a)da \# 5p(22.5)$$

$$K_{20\text{-}24} = \int_{20}^{25} K(a)da \# 5K(22.5)$$

$$C_{20\text{-}24} = \int_{20}^{25} C(a)da \# 5C(22.5)$$

$$D_{20\text{-}24} = \int_{20}^{25} D(a)da \# 5D(22.5)$$

$$d_{20\text{-}24} = \int_{20}^{25} d(a)da \# 5d(22.5)$$

$$B_{20\text{-}24} = \int_{20}^{25} B(a)da \# 5B(22.5)$$

It may be noted in passing that it follows from the first equation that:

$$\overset{0}{e_0} = \sum_0^\omega L_a$$

Morevoer (again disregarding the extremities of life), we have the approximate relationship:

$$\frac{p(22.5) + p(27.5)}{2} \# p(25) \# \frac{1}{10}[L_{20\text{-}24} + L_{25\text{-}29}]$$

It follows that:

$$\overset{25}{e_0} \# \frac{10\sum_{25}^\omega L_a}{L_{20\text{-}24} + L_{25\text{-}29}}$$

(*d*) As the rates are the quotient of two magnitudes, in the process the age-group interval k vanishes. The age-group rates are therefore equal to the instantaneous ones which correspond to the median age of the age group, and we can write:

$$m_{20\text{-}24} = \frac{D_{20\text{-}24}}{K_{20\text{-}24}} \# \frac{5D(22.5)}{5K(22.5)} = \frac{D(22.5)}{K(22.5)} = q(22.5)$$

$$f_{20\text{-}24} = \frac{B_{20\text{-}24}}{K_{20\text{-}24}} \# \frac{5B(22.5)}{5K(22.5)} = \frac{B(22.5)}{K(22.5)} = \varphi(22.5)$$

SOME CONFUSIONS TO BE AVOIDED

The use of the letter "m" may cause confusion, and its meaning needs to be made quite clear:

(*a*) Used alone, this letter designates masculinity at birth;

(*b*) Used as a subscript, it indicates that the function in question refers to the male population;

(*c*) Used as a notation with subscripts, it designates the death rate for a certain age group. For example, when it is used to designate the death rate for the 20-24 age group it is written: $m_{20\text{-}24}$.

The letter f or F may also cause confusion:

(*a*) When f is used as subscript, it indicates that the function in question refers to the female population;

(*b*) When f is used as a notation with subscripts, it designates the fertility rate for a certain age group. For example, the fertility rate for the 20-24 age group is written: $f_{20\text{-}24}$;

(*c*) When F is used as a functional symbol, it designates the age distribution of fertility rates: $F(a)$.

(*d*) When F is used as a symbol with subscripts, it designates the distribution of fertility rates by age groups. For example, the distribution of birth rates for the 20-24 age group is written: $F_{20\text{-}24}$.

FUNDAMENTAL FORMULAE FOR THE FEMALE POPULATION

The main formulae have already been given in chapter I. They are given again in this chapter in various equivalent forms, so that the form best adapted to the problem under consideration can be selected. Some additional formulae not included in chapter I are also given.

The formulae are given in continuous notation. They can easily be written in discontinuous notation by using the approximate relationships between the two systems of notation given above.

Finally, while most formulae are valid only for Malthusian populations, some are valid for all populations, and these are indicated in the course of the text.

(1) The crude female birth rate is written:

$$b_f = \frac{1}{\int_0^\infty e^{-ra} p_f(a)\,da} \qquad \text{(II.1)}$$

(2) The crude female death rate is:

$$d_f = b_f - r \qquad \text{(II.2)}$$

(3) The age distribution of the female population is written:

$$C_f(a) = b_f e^{-ra} p_f(a) \qquad \text{(II.3}a\text{)}$$

This formula can be written in the following equivalent forms:

$$C_f(a) = \frac{e^{-ra} p_f(a)}{\int_0^\infty e^{-ra} p_f(a)\,da} \qquad \text{(II.3}b\text{)}$$

$$C_f(a) = C_f(0) e^{-ra} p_f(a) \qquad \text{(II.3}c\text{)}$$

$$\frac{C_f(a) e^{ra}}{C_f(0)} = p_f(a) \qquad \text{(II.3}d\text{)}$$

Finally, for $a = 0$ we have:

$$C_f(0) = b_f \qquad \text{(II.3}e\text{)}$$

This last formula is valid for all populations.

(4) If we differentiate formula II.3*d* we can write: and consequently:

$$p'_f(a) = \frac{C'_f(a) e^{ra} + r C_f(a) e^{ra}}{C_f(0)}$$

and consequently:

$$q_f(a) = -\frac{p'_f(a)}{p_f(a)} = -\frac{C'_f(a) + r C_f(a)}{C_f(a)}$$

whence finally:

$$q_f(a) = -\frac{C'_f(a)}{C_f(a)} - r = Q_f(a) - r \qquad \text{(II.4)}$$

(5) At time t we have the following formulae for the absolute number of female births $B_f(t)$ and the absolute number of persons in the female population $N_f(t)$:

$$B_f(t) = A e^{rt} \qquad \text{(II.5}a\text{)}$$

$$N_f(t) = \frac{B_f(t)}{b_f} = A e^{rt} \int_0^\infty e^{-ra} p_f(a)\,da \qquad \text{(II.5}b\text{)}$$

where A is a constant equal to the absolute number of female births at time zero (i.e., $B_f(0) = A$).

(6) The age distribution of female deaths is given by the formula:

$$d_f(a) = \frac{C_f(a) q_f(a)}{\int_0^\infty C_f(a) q_f(a)\,da} = \frac{C_f(a) q_f(a)}{d_f} \qquad \text{(II.6}a\text{)}$$

This formula can be written in the following equivalent forms:

$$q_f(a) = \frac{d_f(a)}{C_f(a)} \int_0^\infty C_f(a) q_f(a)\,da = \frac{d_f(a)}{C_f(a)} d_f \qquad \text{(II.6}b\text{)}$$

or:

$$q_f(a) = \frac{d_f(a)}{C_f(a)} [C_f(0) - r] \qquad \text{(II.6}c\text{)}$$

Formulae II.6a, II.6b *and* II.6c *are valid for all populations.*

(7) From formula II.6*a* we deduce that:

$$C_f(a) = \frac{d_f(a)}{q_f(a)} d_f$$

whence, as $C_f(a)$ is a distribution:

$$1 = \int_0^\omega C_f(a)da = d_f \int_0^\omega \frac{d_f(a)}{q_f(a)} da$$

whence:

$$d_f = \frac{1}{\displaystyle\int_0^\omega \frac{d_f(a)}{q_f(a)} da}$$

If we transfer this value of d_f into II.6*b* we obtain:

$$C_f(a) = \frac{\dfrac{d_f(a)}{q_f(a)}}{\displaystyle\int_0^\omega \frac{d_f(a)}{q_f(a)} da} \qquad \text{(II.7)}$$

This last formula is valid for all populations.

(8) In a Malthusian population, the age distribution of deaths is also written:

$$d_f(a) = \frac{e^{-ra} p_f(a) q_f(a)}{\int_0^\omega e^{-ra} p_f(a) q_f(a) da} \qquad \text{(II.8)}$$

(9) As $p_f(a)q_f(a) = -p'_f(a)$, the formula II.8 is written:

$$\frac{d_f}{b_f} d_f(a)e^{ra} = -p'_f(a) \qquad \text{(II.9)}$$

(10) This last formula allows us to write, by integrating:

$$\frac{d_f}{b_f}\int_0^a d_f(a)e^{ra}da = -\int_0^a p'_f(a)da = 1 - p_f(a)$$

If we make $a = \omega$ in this formula, we have:

$$\frac{d_f}{b_f}\int_0^\omega d_f(a)e^{ra}da = 1$$

Whence, by dividing the last two formulae member by member, we have:

$$\frac{\int_0^a d_f(a)e^{ra}da}{\int_0^\omega d_f(a)e^{ra}da} = 1 - p_f(a)$$

whence finally:

$$p_f(a) = 1 - \frac{\int_0^a d_f(a)e^{ra}da}{\int_0^\omega d_f(a)e^{ra}da} \qquad \text{(II.10)}$$

From this value of $p_f(a)$ it is possible to obtain a useful formula for $q_f(a)$.

We have:

$$p'_f(a) = -\frac{d_f(a)e^{ra}}{\int_0^\omega d_f(a)e^{ra}da}$$

whence:

$$q_f(a) = -\frac{p'_f(a)}{p_f(a)} = -\frac{d_f(a)e^{ra}}{\int_0^\omega d_f(a)e^{ra}da - \int_0^a d_f(a)e^{ra}da}$$

$$= -\frac{d_f(a)e^{ra}}{\int_a^\omega d_f(a)e^{ra}da} \qquad \text{(II.10}a\text{)}$$

(11) If we write formulae II.4 and II.6*c* side by side, whe have:

$$q_f(a) = -\frac{C'_f(a)}{C_f(a)} - r$$

$$q_f(a) = \frac{d_f(a)}{C_f(a)}[C_f(0) - r]$$

If we eliminate r from these two equations we obtain:

$$q_f(a) = \frac{d_f(a)}{C_f(a)} \times \frac{C'_f(a) + C_f(a)C_f(0)}{C_f(a) - d_f(a)} \qquad \text{(II.11)}$$

(12) If instead of eliminating r we eliminate $q_f(a)$, we obtain:

$$r = \frac{C_f(0)d_f(a) + C'_f(a)}{-C_f(a) + d_f(a)} \qquad \text{(II.12)}$$

(13) At a given time, we have the equation:

$$\int_u^v e^{-ra} p_f(a) \varphi_f(a, t) da = 1 \qquad \text{(II.13)}$$

In the particular case of the stable population, we assume a value of $\varphi_f(a)$ for the function $\varphi_f(a, t)$ which no longer depends on the time t. The intrinsic rate of natural variation is the real root of the r equation:

$$\int_u^v e^{-ra} p_f(a) \varphi_f(a) da = 1 \qquad \text{(II.14)}$$

This equation differs from equation II.13 only in that the fertility function no longer depends on time ($\varphi_f(a)$ instead of $\varphi_f(a, t)$).

FORMULAE FOR THE MALE POPULATION

All these formulae are valid both for the male population and for populations of both sexes together, provided that the female mortality functions are replaced by the male mortality functions or by the functions for both sexes together, as the case may be.

For equation II.13, $\varphi_f(a, t)$ must be replaced by $\varphi_m(a, t)$, which represents the number of boys fathered by men of age a, while the limits u and v of the reproductive period must be modified. When both sexes are taken together, such an equation obviously becomes meaningless.

Formula II.14 can also be written for male populations. Values cannot be assumed for the functions $\varphi_f(a)$ and $\varphi_m(a)$ simultaneously, however.

If we assume a value for $\varphi_f(a)$, then equation (II.14) determines the intrinsic rate of natural variation ϱ_1 and, consequently, the corresponding stable population. $\varphi_m(a)$ must then satisfy the condition:

$$\int_0^\infty e^{-\varrho_1 a} p_m(a)\varphi_m(a)da = 1$$

Thus, we can no longer assume an arbitrary value for $\varphi_m(a)$.

Likewise, *if we assume a value for* $\varphi_m(a)$, then the equation:

$$\int_0^\infty e^{-ra} p_m(a)\varphi_m(a)da = 1$$

determines an intrinsic rate of natural variation ϱ_2. The function $\varphi_f(a)$ must then satisfy the condition:

$$\int_0^\infty e^{-\varrho_2 a} p_f(a)\varphi_f(a)da = 1$$

Thus, we can no longer assume an arbitrary value for $\varphi_f(a)$.[1]

Hereafter, any mention of a stable population will be a reference to the case where the female fertility function is given.

Table II.2 sets out the various formulae considered above and classifies them according to the parameter which they make it possible to calculate. In order to make them simpler to write, they have been given with neither female nor male subscripts, except in the case of the gross and net reproduction rates, which generally speaking are always for the female sex.

At the end of the list, we have added the traditional formulae giving the gross and net reproduction rates and the approximate formula described in chapter I linking the crude male and female death and birth rates with the same rates for both sexes together.

The formulae in table II.2 are given in continuous notation. They can easily be written in discontinuous notation, but their use then raises new problems which will be considered in connexion with each particular application.

TABLE II.2. BASIC FORMULAE LINKING THE VARIOUS CHARACTERISTICS OF MALTHUSIAN POPULATIONS

Crude birth rate		
	$b = \dfrac{1}{\int_0^\infty e^{-ra}p(a)da}$	(II.1)
	$b = C(0)$	(II.3*e*)*
Crude death rate		
	$d = b - r$	(II.2)
Age structure of the population		
	$C(a) = be^{-ra}p(a)$	(II.3*a*)
	$C(a) = C(0)e^{-ra}p(a)$	(II.3*c*)
	$C(a) = \dfrac{e^{-ra}p(a)}{\int_0^\infty e^{-ra}p(a)da}$	(II.3*b*)
	$C(a) = \dfrac{\frac{d(a)}{q(a)}}{\int_0^\infty \frac{d(a)}{q(a)}da}$	(II.7)
	$C(a) = \dfrac{d(a)}{q(a)} \times d$	(II.6*a*)
Instantaneous death rate		
	$q(a) = -\dfrac{C'(a)}{C(a)} - r = Q(a) - r$	(II.4)
	$q(a) = \dfrac{d(a)}{C(a)}[C(0) - r]$	(II.6*c*)*
	$q(a) = \dfrac{d(a)}{C(a)} \times d$	(II.6*b*)*
	$q(a) = -\dfrac{d(a)e^{ra}}{\int_a^\infty d(a)e^{ra}da}$	(II.10*a*)

---

[1] In an actual population, if $p_f(a)$, $p_m(a)$, $\varphi_f(a)$ and $\varphi_m(a)$ are the mortality and fertility functions *observed* respectively at a given time, the two integral equations:

$$\int_0^\infty e^{-ra}p_m(a)\varphi_m(a)da = 1 \quad \text{and} \quad \int_0^\infty e^{-ra}p_f(a)\varphi_f(a)da = 1$$

are generally incompatible.

TABLE II 2 *(continued)*

$$q(a) = \frac{d(a)}{C(a)} \times \frac{C'(a) + C(a)C(0)}{C(a) - d(a)} \qquad \text{(II.11)}$$

Survivorship function

$$p(a) = \frac{C(a)}{C(0)} e^{ra} \qquad \text{(II.3}d\text{)}$$

$$p(a) = 1 - \frac{\int_0^a d(a)e^{ra}da}{\int_0^\omega d(a)e^{ra}da} \qquad \text{(II.10)}$$

Age structure of deaths

$$d(a) = \frac{C(a)q(a)}{\int_0^\omega C(a)q(a)da} \qquad \text{(II.6}a\text{)}$$

$$d(a) = -\frac{b}{d}\, p'(a)e^{-ra} \qquad \text{(II.9)}$$

$$d(a) = \frac{e^{-ra}p(a)q(a)}{\int_0^\omega e^{-ra}p(a)q(a)da} \qquad \text{(II.8)}$$

Approximate relation between the crude female and crude male birth rates

$$b \,\#\, \frac{b_m + b_f}{2}$$

Approximate relation between the crude female and crude male birth rates

$$d \,\#\, \frac{d_m + d_f}{2}$$

Relation among mortality, fertility and rate of natural increase

$$\int_u^v e^{-ra}p(a)\varphi(a, t)da = 1 \qquad \text{(II.13)}$$

In a stable population, $\varphi(a, t)$ is not dependent on time, and equation (II.13) is written:

$$\int_u^v e^{-ra}p(a)\varphi(a)da = 1 \qquad \text{(II.14)}$$

In addition, we have:

$$\int_u^v \varphi(a, t)C(a)da = b \qquad \text{(II.15)}$$

Gross reproduction rate

$$R' = \int_u^v \varphi_f(a, t)da$$

Net reproduction rate

$$R_0 = \int_u^v p_f(a)\varphi_f(a, t)da$$

Rate of natural variation

$$r = \frac{C(0)d(a) + C'(a)}{-C(a) + d(a)} \qquad \text{(II.12)}$$

Births at time t

$$B(t) = Ae^{rt} = B(0)e^{rt} \qquad \text{(II.5}a\text{)}$$

Population at time t

$$N(t) = B(0)e^{rt}\int_0^\omega e^{-ra}p(a)da \qquad \text{(II.5}b\text{)}$$

Increase at time t

$$rN(t) = rB(0)e^{rt}\int_0^\omega e^{-ra}p(a)da$$

Deaths at time t

$$D(t) = B(0)e^{rt}\left[1 - r\int_0^\omega e^{-ra}p(a)da\right]$$

(*) Formulae valid for all populations.

B. Determination of a population in a sub-set $H_0(r)$ (a Malthusian population whose mortality function is known and which satisfies a given condition)

It has already been stated on several occasions that in a sub-set $H_0(r)$ the knowledge of an additional condition makes it possible to determine a population[2] which satisfies that condition. The principle of this determination is that the additional condition makes it possible to write an equation in r, generally with only one real solution. The population of the sub-set corresponding to that solution[3] is the population satisfying the given condition. We now propose to show, in actual cases, how populations of a sub-set $H_0(r)$ which satisfy a condition are determined.

FIRST EXAMPLE:
THE RATE OF NATURAL VARIATION IS KNOWN

The simplest way of determining a population of a sub-set $H_0(r)$ is obviously to take the rate of natural variation, $r = r_0$. Here is an example of the computations:

Let us consider the sub-set $H_0(r)$ associated with the model life tables given in *Manual III*, which correspond to an expectation of life at birth for both sexes of 50 years (level 60). Let us assume that $r = r_0 = 0.03$ and let us calculate the characteristics of the *female population* which has this rate of increase.

Table II.3 gives details of the computation of the age distribution of the population and the age distribution of deaths.

Age distribution of the population

Let us suppose that we know the survivorship function, in discontinuous notation, for the following age groups: under 1 year, 1-4 years, 5-9 years, and so on by five-year age groups up to 80-84 years, and the 85 and over age group. The corresponding quantities L_a are given in the third column of table II.3.[4] The age distribution of the population is given in continuous notation by formula II.3*a* in table II.2:

$$C_f(a) = b_f e^{-ra} p_f(a) \qquad \text{(II.3}a\text{)}$$

For an age group a, $a + 5$ we assume that:

$$\int_a^{a+5} C_f(a)da = b_f \int_a^{a+5} e^{-ra} p_f(a)da = b_f e^{-r\alpha} L_a$$

where α is the median age of the age group (first column of the table).

The quantities $e^{-r\alpha}$ are given in the fourth column. By multiplying together the third and fourth columns we obtain the quantities $e^{-r\alpha}L_a$ (column 5). The distribution of these quantities is the distribution sought C_a (column 6).

Crude birth rate

The crude birth rate is given in continuous notation by formula II.1 in table II.2:

---

[2] Or at least a small number of populations.

[3] It may happen that the r equation which enables the additional condition to be written has several real roots, but there are never many of them. A population of the set $H(r)$ corresponds to each of these real roots.

[4] These quantities are taken from table IV in the appendix to *Manual III*. See *Manual III: Methods for Population Projections by Sex and Age* (United Nations publication, Sales No.: 56.XIII.3).

TABLE II.3. COMPUTATION OF THE AGE DISTRIBUTION OF THE FEMALE POPULATION AND OF FEMALE DEATHS IN A STABLE POPULATION CORRESPONDING TO AN INTERMEDIATE MODEL LIFE TABLE WITH AN EXPECTATION OF LIFE AT BIRTH FOR BOTH SEXES OF 50 YEARS AND AN INTRINSIC RATE OF NATURAL VARIATION OF 0.03

Median age α *(1)*	*Age group* (a) *(years)* a *(2)*	*Stationary population* L_a *(3)*	$e^{-r\alpha}$ *for* $r = 0.03$ *(4)*	*Product of the two preceding columns* $e^{-r\alpha}L_a$ *(5)*	*Age distribution* C_a *(6)*	*Death rate (per thousand)* m_a *(7)*	*Product of the two preceding columns (deaths) (8)*	*Age distribution of deaths* d_a *(9)*
0.5	0	90 719	0.98511	89 368	40 282	136.41	5 495	366 651
3.0	1-4	338 974	0.91393	309 799	139 640	16.19	2 261	150 864
7.5	5-9	406 628	0.79852	324 701	146 358	3.99	584	38 967
12.5	10-14	399 620	0.68729	274 655	123 799	2.96	366	24 421
17.5	15-19	392 370	0.59156	232 110	104 623	4.38	458	30 560
22.5	20-24	382 368	0.50916	194 686	87 754	5.97	524	34 964
27.5	25-29	370 680	0.43824	162 447	73 222	6.45	472	31 494
32.5	30-34	358 600	0.37719	135 260	60 968	6.80	415	27 691
37.5	35-39	346 202	0.32465	112 394	50 661	7.28	369	24 621
42.5	40-44	333 118	0.27943	93 083	41 957	8.15	342	22 820
47.5	45-49	318 325	0.24051	76 560	34 509	10.06	347	23 153
52.5	50-54	300 392	0.20701	62 184	28 029	13.22	371	24 755
57.5	55-59	277 922	0.17817	49 517	22 320	18.05	403	26 890
62.5	60-64	248 722	0.15336	38 144	17 193	26.79	461	30 760
67.5	65-69	210 400	0.13199	27 771	12 518	41.19	516	34 430
72.5	70-74	162 220	0.11361	18 430	8 307	65.38	543	36 231
77.5	75-79	108 068	0.09778	10 567	4 763	102.30	487	32 495
82.5	80-84	58 022	0.08416	4 883	2 201	154.48	340	22 686
87.5	85 +	27 443	0.07244	1 988	896	259.56	233	15 547
TOTAL . . .		5 130 792		2 218 547	1 000 000		14 987	1 000 000

(a) a represents the starting age of each age group.

$$b_f = \frac{1}{\int_0^\infty e^{-ra} p_f(a) da} \qquad \text{(II.1)}$$

In discontinuous notation, this formula becomes:

$$b_f = \frac{1}{\sum e^{-ra} L_a}$$

In the last line of table II.3 we have, for every 100,000 persons at birth:

$$\sum e^{-ra} L_a = 2\,218\,547$$

whence

$$b_f = \frac{100\,000}{2\,218\,547} = 0.045\,075$$

Crude death rate

$$d_f = b_f - r = 0.045\,075 - 0.030\,000 = 0.015\,075$$

Age distribution of deaths

In continuous notation, the deaths of persons aged between a and $a + da$ years are given, for a population equal to unity, by:

$$D(a) = C(a)\, q(a)\, da$$

In discontinuous notation, for an age group a, $a + 5$ we have:

$$D_a = \sum_a^{a+5} C_a \cdot q_a$$

Let us assume that this expression is equal to $C_a \cdot q_\alpha$, where α is the median age of the age group, and also that the probability of death at the median age q_α is equal to the death rate for the age group m_a. Finally,

$$D_a = C_a . m_a$$

In the particular case considered here, the rates m_a are given in column 7 of table II.3.[5] The products $D_a = C_a m_a$ are given in the following column (column 8). The distribution of the quantities D_a is the age distribution of deaths sought (column 9).

We have confined ourselves to the computation of the characteristics of the female population. The details of the computation of the characteristics of the male population and of the population of both sexes together will be seen in the next example.

SECOND EXAMPLE: THE FEMALE FERTILITY FUNCTION IS KNOWN (STABLE POPULATION)

This additional condition makes it possible to write the r equation (formula II.14 in table II.2):

$$\int_u^v e^{-ra} p_f(a) \varphi_f(a) da = 1 \qquad \text{(II.14)}$$

As will be seen, for a given sub-set $H_0(r)$ this equation has a single real root ρ. When this root is found, we are brought back to the problem dealt with above, i.e., the problem of computing the characteristics of the population of the sub-set $H_0(r)$, whose rate of natural variation is equal to ρ. *This population is called the stable population corresponding to the laws $p_f(a)$ and $\varphi_f(a)$.*

In order to solve equation II.14, we shall first use a graphic method. Such a method has the advantage that it can be applied in all cases, i.e., whatever the functions $p_f(a)$ and $\varphi_f(a)$. We shall then give numerical methods utilizing approximate formulae, which are valid only because the mortality and fertility functions are those applicable to the human species and thus obey certain rules which ensure that their form is not arbitrary. Finally, we shall describe methods using successive approximations which are also valid for human populations.

Uniqueness of the real root ϱ of the fundamental equation

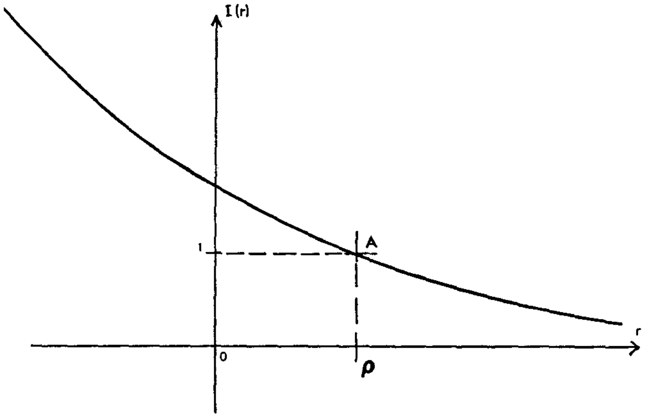
Let us consider the integral

$$I(r) = \int_u^v e^{-ra} p_f(a) \varphi_f(a) da$$

When r approaches $+\infty$, e^{-ra} tends towards zero and $I(r)$ therefore also tends towards zero. When r approaches $-\infty$, e^{-ra} tends towards $+\infty$ and $I(r)$ also tends towards $+\infty$. In addition, we can easily see that $I(r)$ is a decreasing function[6] of r. The curve representing $I(r)$ is thus of the form shown in graph II.1, where r is represented by the horizontal axis and $I(r)$ by the vertical axis.

The straight line of the ordinate equal to unity intersects the curve at one, and only one, point A, whose abscissa ρ is the real root of the equation.

Equation II.14 has therefore only one real root.



Graph II.1. Diagram illustrating the calculation of ϱ by the graphic method

The graphical calculation of ϱ

In order to determine ρ, we must plot the curve representing the integral $I(r)$ and then read off the abscissa of the point of the curve $I(r)$ whose ordinate is equal to unity. By adopting a sufficiently large scale, we can determine ρ with adequate accuracy.

---

[5] These rates are taken from table I in the appendix to *Manual III* (op. cit.).

[6] The derivative $\frac{I(r)}{dr} = -\int_u^v e^{-ra} \cdot a p_f(a) \varphi_f(a) da$ is a negative quantity.

By way of example, we shall calculate the rate ρ corresponding to the mortality and fertility conditions used in chapter I and compute the population projections which, at the limits, approach stable populations. We shall thus be able to verify experimentally that these projections do indeed approach the stable populations which can be computed by solving equation II.14.

The survivorship functions will be those corresponding to the intermediate model life table which provides an expectation of life at birth for both sexes of 60.4 years. We are thus taking a new sub-set, $H_1(r)$, rather than that used in the first example.

The age distribution of female fertility rates will be that described in chapter I. (For reasons which will be given later, this distribution is being called the "intermediate model fertility distribution".) The gross reproduction rates R′ will be equal to 1.5, 0.75 and 1.74675 respectively.[7]

Equation (II.14) is written in continuous notation, but in practice we must write it in discontinuous notation. If we have the data by five-year age groups, then we have:

$$\sum_{15}^{40} e^{-r\alpha} L_a f_a = 1$$

where α is the median age of the successive age groups, i.e. 17.5, 22.5 ... 42.5, and a is the starting point of the age groups.

As we have:

$$R' = 5\sum_{15}^{40} f_a$$

and

$$F_a = \frac{f_a}{\sum_{15}^{40} f_a} = \frac{5f_a}{R'}$$

equation (II.14) is written, in discontinuous notation:

$$\sum_{15}^{40} e^{-ra} L_a F_a = \frac{5}{R'} \qquad \text{(II.14}bis\text{)}$$

Table II.14 gives the computation of the sum:

$$\sum_{15}^{40} e^{-ra} L_a F_a$$

for various values of r and for 100,000 girls at birth. Graph II.2 gives the curve representing this sum as a function of r.

For a gross reproduction rate $R' = 1.5$, we have $5/R' = 3.333$ ..., and the point on the curve having an ordinate of 3.333 ... corresponds to an abscissa $\rho_1 = 0.009$. For a reproduction rate $R' = 1.174675$, we have $5/R' = 4.26$ and the point on the curve corresponding to an ordinate of 4.26 for abscissa[7] $\rho_2 = 0$.

Finally, for a gross reproduction rate $R' = 0.75$, we have $5/R' = 6.666$ ..., and the point on the curve having an ordinate of 6.666 ... corresponds to an abscissa $\rho_3 = 0.017$. For computations of this nature, we are constantly in need of the values of the function e^{-ra}

TABLE II.4. COMPUTATION OF THE SUM $\sum_{15}^{40} e^{-ra} L_a F_a$ FOR VARIOUS VALUES OF r

Conditions: intermediate model life table corresponding to an expectation of life at birth for both sexes of 60.4 years, and intermediate model fertility distribution

Age group (years) (1)	*Median age* a *(2)*	*Female stationary population* (L_a) *(per 100,000 births) (3)*	*Age distribution of female fertility rates* (F_a) *(4)*	*Product of the two preceding columns (5)*
15-19	17.5	439 970	0.100	43 997
20-24	22.5	434 040	0.273	118 493
25-29	27.5	427 035	0.263	112 310
30-34	32.5	419 610	0.188	78 887
35-39	37.5	411 672	0.121	49 812
40-44	42.5	402 742	0.055	22 151

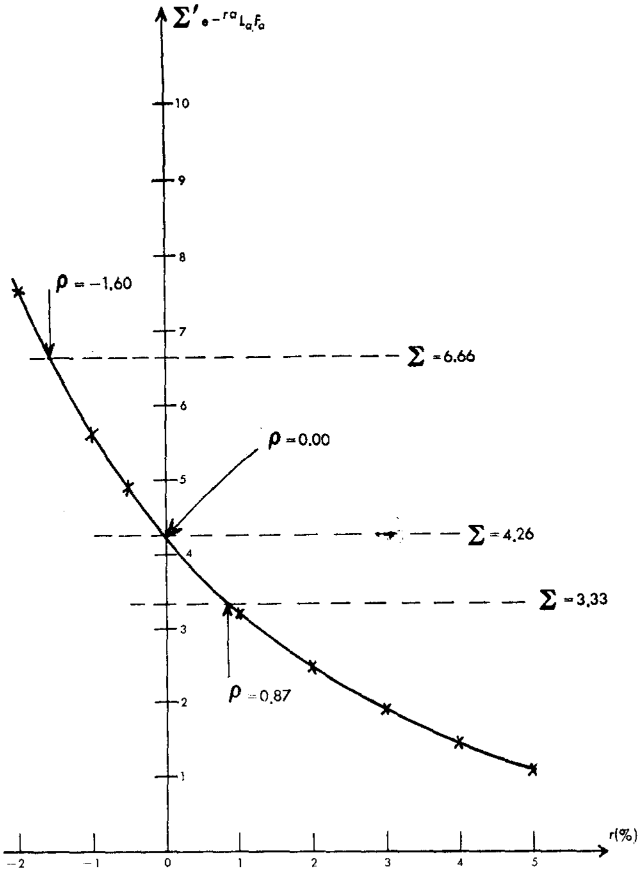
Product of the multiplication of column (5) by e^{-ra} *for the following values of* r *(percentages)*

Age group (years)	*5*	*4*	*3*	*2*	*1*	*0.5*	*0.0*	*−0.5*	*−1.0*	*−2.0*
15-19 . . .	18 341	21 848	26 027	31 005	36 934	40 311	43 997	48 064	52 412	62 484
20-24 . . .	38 469	48 176	60 332	75 555	94 619	105 885	118 493	132 614	148 391	185 834
25-29 . . .	28 396	37 385	49 219	64 797	85 307	97 882	112 310	128 864	147 860	194 673
30-34 . . .	15 534	21 499	29 755	41 183	56 998	67 055	78 887	92 807	109 182	151 111
35-39 . . .	7 639	11 115	16 171	23 530	34 236	41 295	49 812	60 085	72 476	105 453
40-44 . . .	2 645	4 047	6 190	9 469	14 482	17 910	22 151	27 865	33 882	51 812
TOTAL . . .	111 024	144 070	187 694	245 539	322 575	370 339	425 650	490 299	564 203	751 367

---

[7] It will be remembered that the last-mentioned value is such that the corresponding stable population is identical with the stationary population (i.e., a population whose rate of natural increase is zero).

for various values of r and for the traditional five-year age groups. These can be calculated once for all, as has been done in tables II.5 and II.6.



Graph II.2. Curve showing variation of the sum $\sum_{15}^{40} e^{-ra}L_aF_a$ as a function of r (the figures in the last line of table II.4, divided by 100,000)

At the scale used in graph II.2, the accuracy achieved is not very great. It could obviously be improved by enlarging the scale, but approximate formulae can be established; these will be dealt with next.

Practical limits of the variation of the intrinsic rate

In practice, the intrinsic rate of natural variation varies in the human species only within quite narrow limits. If, in the preceding formulae, we replace the various functions of r by their expansions in an increasing-power series in r, we can, because of the smallness of r, disregard the terms of these expansions beyond a certain order and establish approximate numerical formulae which enable us to compute the various characteristics of a stable population. It is therefore essential to determine exactly the practical limits of variation in the intrinsic rate of natural variation.

If we assume that no women are sterile between the ages of 15 and 45 and that each woman gives birth to one child every year, the female fertility rate is constant from the age of 15 to the age of 45 and is approximately[8] equal to 0.5

If we also assume that mortality is nil up to the age of 45, we have $p_f(a) = 1$, and equation (II.14) is written:

$$\int_u^v e^{-ra}\, da = 2$$

The real root of this equation is very close to $\rho = 0.10$. Thus, a rate of increase of 10 per cent appears to be the upper limit of the intrinsic rate of natural variation. In fact, however, there are women who have no children, either because they are sterile or because they are unmarried. Moreover, even fertile women do not conceive

TABLE II.5. VALUES OF e^{-ra} FOR VARIOUS VALUES OF a AND r

Age group (years)	*Median age* a	*Intrinsic rate of natural variation* r *(percentage)*											
		−2.0	*−1.5*	*−1.0*	*−0.5*	*+0.5*	*+1.0*	*+1.5*	*+2.0*	*+2.5*	*+3.0*	*+3.5*	*+4.0*
0 . . .	0.5	1.01000	1.00750	1.00500	1.00250	0.99005	0.99181	0.99501	0.99750	0.98758	0.98511	0.98236	0.98020
1-4 . .	3.0	1.06180	1.04600	1.03050	1.01510	0.98511	0.97044	0.95600	0.94176	0.92774	0.91393	0.90033	0.88691
0-4 . .	2.5	1.05137	1.03821	1.02532	1.01268	0.98758	0.97531	0.96319	0.95123	0.93941	0.92774	0.91622	0.90482
5-9 . .	7.5	1.16283	1.11917	1.07788	1.03821	0.96319	0.92774	0.89360	0.86071	0.82901	0.79852	0.76913	0.74082
10-14 .	12.5	1.28402	1.20623	1.13315	1.06559	0.93941	0.88250	0.82903	0.77880	0.73162	0.68729	0.64565	0.60653
15-19 .	17.5	1.42017	1.30128	1.19125	1.09244	0.91622	0.83946	0.76913	0.70469	0.64565	0.59156	0.54199	0.49659
20-24 .	22.5	1.56831	1.40144	1.25232	1.11917	0.89360	0.79852	0.71355	0.63763	0.56977	0.50916	0.45498	0.40657
25-29 .	27.5	1.73335	1.51059	1.31653	1.14740	0.87153	0.75957	0.66199	0.57695	0.50283	0.43824	0.38194	0.33287
30-34 .	32.5	1.91554	1.62829	1.38403	1.17645	0.85002	0.72253	0.61416	0.52205	0.44375	0.37719	0.32061	0.27253
35-39 .	37.5	2.11700	1.75515	1.45499	1.20623	0.82903	0.68729	0.56978	0.47237	0.39160	0.32465	0.26915	0.22313
40-44 .	42.5	2.33905	1.89174	1.52959	1.25796	0.80856	0.65377	0.52861	0.42742	0.34559	0.27943	0.22594	0.18268
45-49 .	47.5	2.58571	2.03918	1.60801	1.26818	0.78860	0.62189	0.49042	0.38674	0.30498	0.24501	0.18966	0.14957
50-54 .	52.5	2.85805	2.19899	1.69046	1.30128	0.76913	0.59156	0.45498	0.34994	0.26913	0.20701	0.15922	0.12246
55-59 .	57.5	3.15829	2.36918	1.77713	1.33319	0.75014	0.56271	0.42211	0.31664	0.23752	0.17817	0.13365	0.10026
60-64 .	62.5	3.49034	2.55369	1.86824	1.36684	0.73162	0.53526	0.39161	0.28651	0.20961	0.15336	0.11220	0.08209
65-69 .	67.5	3.85743	2.75257	1.96403	1.40144	0.71355	0.50916	0.36331	0.25924	0.18508	0.13199	0.09418	0.06721
70-74 .	72.5	4.26311	2.96685	2.06473	1.43092	0.69593	0.48432	0.33706	0.23457	0.16235	0.11361	0.07906	0.05502
75-79 .	77.5	4.71157	3.19192	2.17059	1.47339	0.67875	0.46070	0.31270	0.21225	0.14406	0.09778	0.06637	0.04505
80-84 .	82.5	5.20708	3.44709	2.28188	1.51169	0.66199	0.43824	0.29011	0.19205	0.12714	0.08416	0.05572	0.03688
85 + .	87.5	5.75560	3.71665	2.39887	1.51705	0.64565	0.41686	0.26915	0.17377	0.11220	0.07244	0.04677	0.03020

Note: $e = 2.7182818$; Log $e = 4342945$.

---

[8] We say "approximately" because more boys than girls are born; consequently, one child born every year would mean slightly less than 0.5 girls per year.

at the rate of one child per year; the interval between births is generally about 2.5 years. Finally, the mortality rate up to the age of 45 will never be nil. Thus, the limit of 10 per cent is very liberally calculated; indeed, rates even as high as 4 per cent are very seldom encountered in practice.

Where negative intrinsic rates of natural variation are concerned, there may well be cases of very rapid population decline. Examples could be found of rapid population decline in certain cities[9] or certain sub-groups of the population. However, an intrinsic rate of natural variation of −10 per cent would mean an extremely rapid natural decrease, for in twenty years the population would decline by 93 per cent.

In the final analysis, then, it seems that if we assume that intrinsic rates of natural variation always remain between −10 per cent and +10 per cent we are sure to encompass nearly all actual populations.

It is frequently necessary, in this study, to seek to determine the form of the curves representing various functions of r, and perhaps to examine how they behave for values of r far removed from the range of −0.10 to +0.10 or even to consider what happens to the functions in the case of infinitely great values of r, as in the case of the integral $I(r)$ above. This obviously has no practical significance; the only object of such computations is to permit a more accurate determination of the form of the curves, and only that part of the curves corresponding to values of r between −0.10 and +0.10 is of any significance.

TABLE II.6. VALUES OF e^{-ra} FOR VARIOUS VALUES OF a AND TWO VALUES OF r ($r = 0.01$ and $r = -0.01$)

Age group (years)	*Median age* a	*Intrinsic rate of natural variation* r *(percentage)*	
		−10	*+10*
0	0.5	1.0513	0.95123
1-4	3.0	1.3191	0.74330
0-4	2.5	1.2840	0.77880
5-9	7.5	2.1170	0.47237
10-14	12.5	3.4903	0.28650
15-19	17.5	5.7546	0.17377
20-24	22.5	9.4877	0.10540
25-29	27.5	15.643	0.06393
30-34	32.5	25.790	0.03877
35-39	37.5	42.521	0.02352
40-44	42.5	70.105	0.01426
45-49	47.5	115.58	0.00865
50-54	52.5	190.57	0.00525
55-59	57.5	314.19	0.00318
60-64	62.5	518.01	0.00193
65-69	67.5	854.06	0.00117
70-74	72.5	1 408.1	0.00071
75-79	77.5	2,321.6	0.00043
80-84	82.5	3 827.6	0.00026
85 and over . .	87.5	6 310.7	0.00016

Approximate numerical formulae for the computation of the intrinsic rate of natural variation

There are, of course, various ways of establishing approximate formulae. The method proposed by Lotka involves the use of expansion in series of the moments R_n and the cumulants μ_n of a function $f(x)$. The moment R_n is defined by the integral

$$R_n = \int_0^{+\infty} x^n f(x)dx$$

while the cumulants $\mu_1, \mu_2, \dots, \mu_n$ are deduced from the moments R by the solution of the following equations:

$$\begin{aligned} R_1 &= \mu_1 R_0 \\ R_2 &= \mu_1 R_1 + \mu_2 R_0 \\ R_3 &= \mu_1 R_2 + 2\mu_2 R_1 + \mu_3 R_0 \\ R_4 &= \mu_1 R_3 + 3\mu_2 R_1 + 3\mu_3 R_0 + \mu_4 R_0 \\ &\text{etc.} \end{aligned}$$

The coefficients of the successive equations are those of Newton's binomial formula.

First approximation

This is obtained by disregarding all the terms above the first order. It is written:[10]

$$\varrho = \frac{\text{Log } R_0}{\mu_1 \times 0.4342945} = \text{Log } R_0 \times \frac{2.302584}{\mu_1} \qquad \text{(II.16)}$$

where R_0 is the zero-order moment of the function $p_f(a)m_f(a)$ and μ_1 is the first-order cumulant of the same function. The expression:

$$R_0 = \int_u^v p_f(a)\varphi_f(a)da$$

gives us the net reproduction rate, and

$$\mu_1 = \frac{R_1}{R_0} = \frac{\int_u^v a p_f(a)\varphi_f(a)da}{\int_u^v p_f(a)\varphi_f(a)da}$$

is therefore the average age of mothers at the birth of their children *in the stationary population.*

Second approximation

This is obtained by disregarding all the terms above the second order:

$$\varrho = \frac{\mu_1 \pm \sqrt{\mu_1^2 - 2\mu_2 \text{ Log } R_0 \times 2.302584}}{\mu_2} \qquad \text{(II.17)}$$

where R_0 is always the zero-order moment of the function $p_f(a)\varphi_f(a)$ and μ_1 and μ_2 are the first-order and second-order cumulants of the same function.[11]

---

[9] This refers to natural variation, and not to total variation (i.e., it does not include the effects of migratory movement). A city with very low fertility may have a very high negative intrinsic rate of natural variation, and yet its population may not decrease, owing to extensive immigration.

[10] The numerical coefficient in the denominator of this formula is simply: Log e = 0.4342945.

[11] We select for placing before the root symbol the symbol which gives a value close to that given by the first approximate formula.

Tables II.7 and II.8 give details of the application of these two formulae in the three cases so far considered.

The last line of table II.8 gives the values of r obtained by the geometric method. It will be seen that, for the two values of the gross reproduction rate used here (1.50 and 0.75), the three methods give very similar results. In order to obtain at least an idea of the maximum value of the gross reproduction rate for which the approximation formula can be used, we have added to the preceding table the results obtained by the three methods using a gross reproduction rate of 4.0. Even at such a high value, the first formula still gives a good approximation. The second formula and the geometric method give practically identical results. There is very little likelihood of ever encountering a human fertility rate which gives a gross reproduction rate of over 4.0. Thus, we see that the approximation formulae are generally quite adequate in practice for the computation of the intrinsic rate of natural variation of stable populations.

Computation of the intrinsic rate of natural variation by the method of successive approximations

Many ways of computing the intrinsic rate of natural variation by the method of successive approximations can be envisaged, but we shall confine ourselves to two of them.

First method (the mean interval between two generations)

The first method involves a notion frequently utilized in demography, namely, that of the mean interval between two generations.

The word "generation" is used in two senses in demography. First, it may mean a group of persons born at the same time—for example, all the persons born during a calendar year. It is also used to mean a group of children in relation to their parents, or *vice versa*. The group of parents is called the first generation, the group of children is called the second generation, and it is easy to see what is meant by third generation, fourth generation, and so on. What, then, does the expression "interval between two generations" mean?

Let us consider a set of girls born at the same time, and let us represent the mean age of their mothers at that time by A. If the risk of death was nil, the mean age of the set of girls under consideration throughout their lives would be A years less than the mean age of their mothers. In fact, however, death exercises an effect, and the difference between the two mean ages declines as the "generation" of girls grows older, although the decline is quite slow.[12] It seems logical to call this mean age A the "interval between two generations". In a stable population it is written:

---

[12] Assuming mortality to be nil, if in a census every woman is asked the age of her mother and the replies are classified according to the age of the daughters, we shall have for each age of the daughters a mean age of the mothers which is greater than the age of the daughters by A years. Mortality affects the results in two ways. In the first place, as the group of daughters grows older, the older mothers die and the difference between the mean age of the mothers and that of the daughters becomes less. This effect can be corrected by asking the daughters the year of birth of their mothers, whether or not they are still alive. However, mortality can also affect the difference A if there is a correlation between the fertility of a mother and the age at death of her daughters. In fact, it is recognized that such a correlation exists. The difference between the mean age of daughters and the mean age of mothers is greater in the lower than in the higher social classes, because of a difference in the fertility of the two classes. Women of the lower social classes continue to have children at relatively advanced ages, while in the higher social classes childbearing ceases quite early. However, mortality is also higher in the lower social classes. The result is that daughters of the lower social classes are represented to an increasingly small extent in the group of daughters questioned as the age of the daughters increases, thus tending to reduce A, although this latter factor exercises its effect only at advanced ages, when mortality becomes high.

TABLE II.7. COMPUTATION OF THE FIRST-ORDER AND SECOND-ORDER MOMENTS OF THE FUNCTION $p_f(a)m_f(a)$ FOR A GROSS REPRODUCTION RATE OF 5.0 (a) (LEVEL-80 MODEL LIFE TABLE AND INTERMEDIATE MODEL FERTILITY DISTRIBUTION)

Age group (years)	*Median age* α	*Zero-order* (b) *moment:* R_0	*First-order* (c) *moment:* R_1	*Second-order* (c) *moment:* R_2
15-19	17.5	43 997	769 948	13 474 090
20-24	22.5	118 493	2 666 093	59 987 092
25-29	27.5	112 310	3 088 575	84 934 438
30-34	32.5	78 887	2 563 828	83 324 410
35-39	37.5	49 812	1 867 950	70 048 125
40-44	42.5	22 151	941 418	40 010 265
TOTAL		425 650	11 897 812	351 778 420
Moments . . .		4.2565	118.9781	3 517.7842

(a) The gross reproduction rate is equal to five times the total of the female fertility rates by five-year age groups. In table II.4, as F_a is a distribution, we have:

$$\sum_{15}^{40} F_a = 1$$

and consequently:

$$5 \sum_{15}^{40} F_a = 5$$

We can therefore consider F_a as the fertility rate by age groups where the gross reproduction rate is equal to 5.

(b) These are the figures from column 5 of table II.4.

(c) These are the figures from the previous column, multiplied by the median age.

$$A(r) = \frac{\int_u^v a e^{-ra} p_f(a)\varphi_f(a)da}{\int_u^v e^{-ra} p_f(a)\varphi_f(a)da}$$

The problem can also be reasoned in the following manner. Let us consider a group of girls all born during the same year. In the course of their reproductive life, they give birth to daughters at various ages. When they reach the end of the reproductive period, we can calculate the mean age B at which they gave birth to their daughters. If the risk of death is nil, there will in the future always be the same difference B between the mean age of the group of mothers under consideration and the mean age of their daughters. In fact, however, mortality occurs, and the difference increases slowly as the group of mothers grows older.[13] It therefore appears to be quite as logical as in the preceding case to call the mean age B the "mean interval between two generations". Its value is:

$$B = \frac{\int_u^v a p_f(a)\varphi_f(a)da}{\int_u^v p_f(a)\varphi_f(a)da}$$

It will be seen that B is different from A_r. More precisely, B is equal to the value assumed by A_r when the population is stationary. We can write: $B = A_0$.

In his theory of stable populations, Lotka gives the name of "interval between two generations" to yet another formula. He considers the integral:

$$I(r) = \int_u^v e^{-ra} p_f(a)\varphi_f(a)da$$

and he writes

$$\frac{dI(r)}{dr} = -\int_u^v a e^{-ra} p_f(a)\varphi_f(a)da$$

If we put:

$$A(r) = \frac{\int_u^v a e^{-ra} p_f(a)\varphi_f(a)da}{\int_u^v e^{-ra} p_f(a)\varphi_f(a)da}$$

we have

$$\frac{dI(r)}{dr} = -(Ar)I(r)$$

whence

$$I(r) = Ke^{-\int_0^r A(r)dr}$$

where K is a constant.

When $r = 0$ we have $I(0) = K = R_0$ (the net reproduction rate). We therefore have:

$$I(r) = R_0 e^{-\int_0^r A(r)dr}$$

---

[13] If a population census is taken, the two aspects referred to in the preceding foot-note will be noted.

TABLE II.8. COMPUTATION OF THE INTRINSIC RATE OF NATURAL VARIATION BY APPROXIMATION

Stage of calculation	*Gross reproduction rate:* R′		
	1.50	*0.75*	*4.00*
Zero-order moment (a) (R_0)	1.27695	0.638475	3.4052
Log R_0	0.1061739	−0.1948561	0.5321426
Cumulant (b) $\mu_1 = \frac{R_1}{R_0}$	27.9251	27.9521	27.9521
Cumulant (b) $\mu_2 = \frac{R_2}{R_1}$	29.5666	29.5666	29.5666
		First approximation	
μ_1 Log $e = \mu_1 \times 0.4342945$.	12.13944	12.13944	12.13944
$r = \frac{\text{Log } R_0}{\mu_1 \text{ Log } e}$.	+0.00874	−0.01605	+0.04384
		Second approximation	
$\frac{2\mu_2}{\text{Log } e} = 2\mu_2 \times 2.302584$.	136.1504	136.1504	136.1504
$\frac{\text{Log } R_0 \times 2\mu_2}{\text{Log } e}$	14.4563	−26.4309	72.4546
$(\mu_1)^2$	781.3199	781.3199	781.3199
Difference between the two preceding lines	766.8636	807.8508	708.8653
Square root of the preceding line	27.6923	28.4228	26.6245
μ_1 less the preceding line.	0.2598	−0.4707	1.3276
The preceding line divided by μ_2	0.00879	−0.01592	+0.04490
Resultats obtained by the geometric method	0.0090	−0.0160	+0.0450

(a) The zero-order moment of table II.7 multiplied by R′/5.

(b) As the cumulants are equal to the ratios between two successive moments, they become independent of the gross reproduction rate as soon as a value is assumed for the age distribution of the fertility rates.

Lotka then puts:

$$\int_0^r A(r)dr = rT_r$$

and finally we write:

$$I(r) = R_0 e^{-rT_r}$$

In the stable population, $I(r) = 1$. Thus, we have:

$$R_0 = e^{rT_r}$$

T_r is therefore the time needed for the stable population to be multiplied by the net reproduction rate. Lotka calls this time T_r the "mean interval between two generations".

Thus, we have three definitions, all of them apparently valid, for the interval between two generations. Fortunately, when r is small there is a very simple approximate relationship between these three formulae.

Let us revert to the equation:

$$A(r) = \frac{\int_u^v a e^{-ra} p_f(a) \varphi_f(a) da}{\int_u^v e^{-ra} p_f(a) \varphi_f(a) da}$$

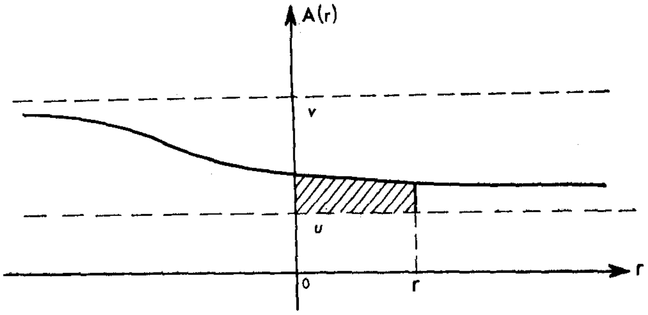
If we introduce the age distribution of fertility rates $F(a)$, this equation is then written:

$$A(r) = \frac{\int_u^v a e^{-ra} p_f(a) F(a) da}{\int_u^v e^{-ra} p_f(a) F(a) da}$$

In order to clarify our ideas, let us now take the case where $F(a)$ does not vary, i.e., where the age distribution of fertility rates remains constant when fertility varies. In such conditions, $A(r)$ is a function only of r, of which it is a decreasing function. In order to obtain the values of $A(r)$ for infinitely great values of r, it is convenient to imagine the formula above written in discontinuous notation. If r tends towards $+\infty$, then the term close to u becomes preponderant over all the others and $A(r)$ tends towards:

$$\frac{u e^{-ru} p_f(u) F(a)}{e^{-ru} p_f(u) F(a)} = u$$

Likewise, if r tends towards $-\infty$ we see that $A(r)$ tends towards V. The curve representing $A(r)$ as a function of r therefore has the form shown in graph II.3.



Graph II.3. Curve representing the variation of $A(r)$ as a function of r

If $r = 0$, then we obviously have $Ar = A_0 = B$. Around $r = 0$, the variations of $A(r)$ as a function of r are almost linear. This can be seen from table II.9, where we have calculated, for seven values of r ranging from 0 to 7 per cent, the values of $A(r)$ computed by associating the level—80 intermediate model life table (giving an expectation of life at birth for both sexes of 60.4 years) with a fertility function of the intermediate model age distribution. The slight variation in the difference computed in the third line of table II.9 clearly shows that $A(r)$ is, practically speaking, a decreasing linear function of r. For other age distributions of the fertility rate and for other model life tables we should have other values of $A(r)$, but we should still find a linear relation between $A(r)$ and r. It was seen above that for $r = 0$ we have $A_0 = B$. We therefore have, as a good approximation:

$$T_r = \frac{1}{r} \int_0^r A_r dr \; \# \; \frac{A_r + A_0}{2}.$$

We see that T_r is also, practically speaking, a linear function of r. In particular, when r tends towards 0, T_r tends towards A_0. The last two lines of the table enable the results of the computation of T_r by approximation to be compared with the results of the exact computation.

Naturally, all these formulae are valid only for values of r close to $r = 0$. As soon as we take very large values of r they no longer hold good.[14]

TABLE II.9. VARIATIONS IN THE MEAN AGE A_r AS A FUNCTION OF r (UNITED NATIONS INTERMEDIATE MODEL LIFE TABLE GIVING AN EXPECTATION OF LIFE AT BIRTH FOR BOTH SEXES OF 60.4 YEARS). INTERMEDIATE MODEL FERTILITY DISTRIBUTION (TABLE I.1 IN CHAPTER I, SECTION D)

Rate r (percentage)	0	1	2	3	4	5	6	7
Age A_r (years)	27.952	27.506	27.074	26.655	26.251	25.861	25.486	25.126
Difference between successive values of A_r	0.446	0.432	0.419	0.404	0.390	0.375	0.360	
Calculation of T_r								
(1) Approximate formula .		27.729	27.509	27.294	27.084	26.878	26.878	26.481
(2) Exact formula		27.729	27.513	27.303	27.102	26.906	26.719	26.539

---

[14] The hatched area on graph II.3 represents the integral

$$\int_0^r A(r)\, dr$$

As r approaches $+\infty$, the integral tends towards ur, and consequently T_r tends towards u. Similarly, when r tends towards $-\infty$, T_r tends towards v. The curve representing T_r has the same asymptotes as the curve representing A_r, and for $r = 0$ we have $T_r = A_r = B_0$. The two curves representing A_r and T_r are very close to each other.

Let us return to the equation:

$$R_0 = e^{rT_r}$$

whence we obtain:

$$r = \frac{\text{Log } R_0}{T_r \text{ Log } e}$$

This is the equation which will be used to calculate r by the method of successive approximations. The value sought, r, is the abscissa of the point of intersection of the two curves:

$$\text{(I)} \qquad T_r = \frac{\text{Log } R_0}{\text{Log } e} \cdot \frac{1}{r}$$

$$\text{and (II)} \qquad T_r = \frac{1}{r}\int_0^r A_r dr$$

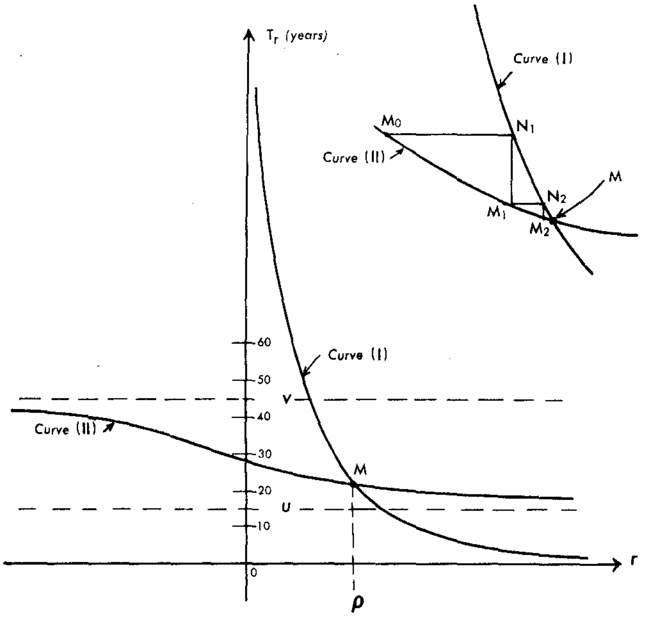
plotted on a graph where r is represented by the horizontal axis and T_r by the vertical. Graph II.4 shows these curves for a gross reproduction rate of 4.00 (the age distribution of fertility being those of the intermediate model) and for an intermediate model life table corresponding to an expectation of life at birth for both sexes of 60.4 years. Curve I is a branch of an equilateral hyperbola, while curve II is of the form shown in graph II.3. The two curves intersect at a point M whose abscissa is the intrinsic rate of natural variation ρ. Let r_0 be an approximate value of r, and let us consider the series:

$$r_1 = \frac{\text{Log } R_0}{T_{r_0} \text{ Log } e} \qquad r_2 = \frac{\text{Log } R_0}{T_{r_1} \text{ Log } e}$$

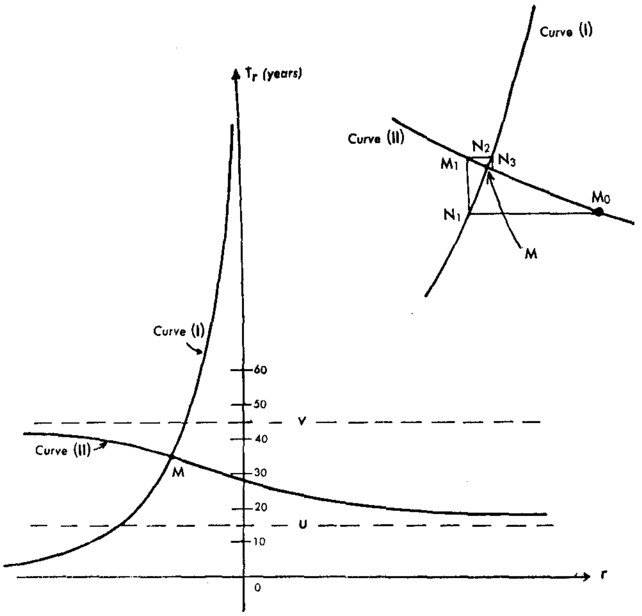
$$r_3 = \frac{\text{Log } R_0}{T_{r_2} \text{Log } e} \text{ etc.}$$

and T_{r_0}, T_{r_1}, T_{r_2}, etc., being calculated by the formula

$$T_r = \frac{1}{r}\int_0^r A_r dr$$



Graph II.4. Graphic illustration of the first method of determining the intrinsic rate of natural variation ϱ by successive approximations (where ϱ is positive)



Graph II.5. Illustration of the first method of determining the intrinsic rate of natural variation ϱ by successive approximations (where ϱ is negative)

Let us begin from the point M_0 of curve II which has as its abscissa r_0, and let us plot the path $M_0N_1M_1N_2M_2$. As the inset in the graph indicates, the points M_0, M_1, M_2, etc., have as their abscissae terms of the series r_0, r_1, r_2, etc. The form of the curves is such that this series tends toward r. If we start from an approximate value which is too low we tend towards r but remain always below it, while if we start from an approximate value which is too high we tend towards r but remain always above it. In this way, we can close in on the true value of r by successive approximations. Curve II is very flat, and consequently the series converges very rapidly on r.

If r is negative (gross reproduction rate of 0.75), the results are slightly different. Graph II.5 shows that we converge successively on the true value of r. If r_0 is smaller than ρ, r_1 will be larger and r_2 will be smaller, and so on. Let us see how this method is applied to an actual example. The model life table is of the intermediate model, giving an expectation of life at birth for both sexes of 60.4 years, while the gross reproduction rate is 4.0. Let us take $r_0 = 0.04$ in our first calculation and $r_0 = 0.05$ in our second. Here are the successive phases of the computation:

(*a*) For a gross reproduction rate $R' = 5.0$, we have $R_0 = 4.25650$ (figures taken from table II.7).

(*b*) For a gross reproduction rate $R' = 4.0$, we have

$$R_0 = \frac{4.25650 \times 4}{5} = 3.4052$$

(figures taken from table II.8).

(*c*) We deduce from this that:

$$\frac{\text{Log } R_0}{\text{Log } e} = 1.225303$$

(*d*) For $r = 0.04$, we have

$$T_r = \frac{A_0 + A_r}{2} = \frac{26.251 + 27.952}{2} = 27.102$$

(*e*) For $r = 0.05$ we have

$$T_r = \frac{A_0 + A_r}{2} = \frac{25.861 + 27.952}{2} = 26.906$$

(In these two computations of T_r, the values of A_0 and A_r are taken from table II.8.)

(*f*) If we start from $r_0 = 0.04$, we have

$$r_1 = \frac{1.225303}{27.102} = 0.0452$$

(*g*) If we start from $r_0 = 0.05$, we have

$$r_1 = \frac{1.225303}{26.906} = 0.0455$$

We ultimately find that the value of r is between 0.0452 and 0.0455. Although the approximate starting values may be very far from the true value, we obtain at the first approximation values very close to the value sought. It should be noted that the first approximate formula used earlier in this section is simply the first term of the above series when we start from $r_0 = 0$.

Second method of successive approximations

Let r_0 be an approximate value of ρ. Let us assume that $r_0 + \varepsilon = \rho$. The basic equation is then written:

$$\int_u^v e^{-(r_0+\varepsilon)a} p_f(a)\varphi_f(a)da = 1$$

or:

$$\int_u^v e^{-r_0 a} p_f(a)\varphi_f(a) e^{-\varepsilon a}\, da = 1$$

and, as ε is small, we can write:

$$\int_u^v e^{-r_0 a} p_f(a)\varphi_f(a)\,(1 - \varepsilon a)da = 1$$

whence we have:

$$\varepsilon = \frac{\int_u^v e^{-ra} p_f(a)\varphi_f(a)da - 1}{\int_u^v e^{-ra} a p_f(a)\varphi_f(a)da} \qquad \text{(II.18)}$$

$1 - \varepsilon a$ is smaller than $e^{-\varepsilon a}$. If we replace $e^{-\varepsilon a}$ by $1 - \varepsilon a$, we therefore find a value of ε which is too small. The quantity $r_1 = r_0 + \varepsilon$ is a better approximation than r_0. If we begin the computation over again with r_1 we therefore obtain a still better value r_2, and so forth. The graphic interpretation of the method is very simple. On a graph where r is represented on the horizontal axis, the value of r which is sought is the abscissa of the point of intersection of the two curves:

$$\text{(I)} \qquad y = \frac{\int_u^v e^{-ra} p_f(a)\varphi_f(a)da - 1}{\int_u^v a e^{-ra} p_f(a)\varphi_f(a)da} + r$$

$$\text{(II)} \qquad y = r$$

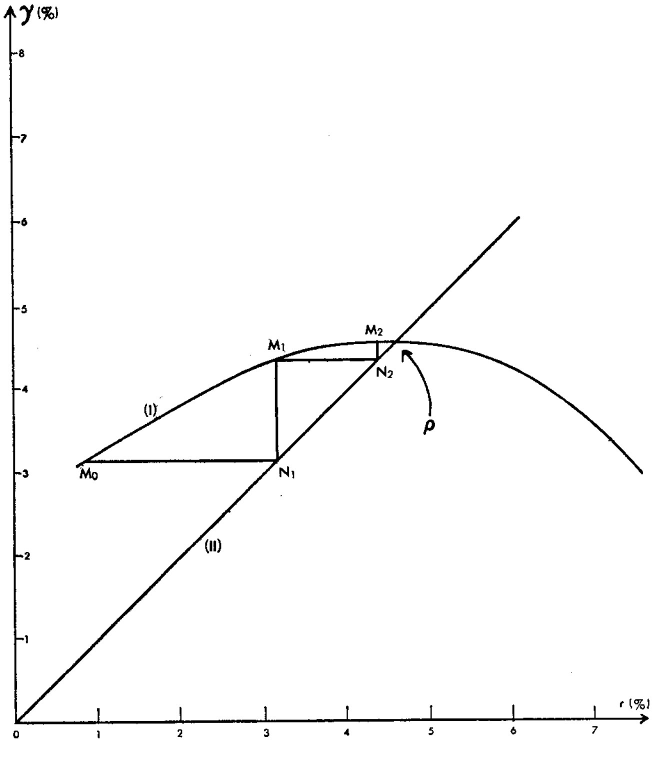
Graph II.6 shows the two curves (I) and (II) corresponding to a gross reproduction rate of 4.00. If, starting from a point on curve (I) with an abscissa r_0, we construct the path $M_0N_1M_1N_2M_2$ we see that the points M_0, M_1, M_2 etc., have as their abscissae r_0, r_1, r_2 etc. Graph II.6 shows that the series converges towards ρ. If we start from a value of r_0 which is too small, we remain consistently below the true value. If we start from a value which is too high, we shall be too low at the first approximation and shall remain so subsequently. Thus, in this method the series always converges towards r by giving values which are too small.

The correction for which a formula has been given above can be put in an approximate form which makes possible a very rapid method of computing ρ. We can write:

$$\varepsilon = \frac{1 - \dfrac{1}{\int_u^v e^{-r_0 a} p_f(a)\varphi_f(a)da}}{\dfrac{\int_u^v e^{-ra} a p_f(a)\varphi_f(a)da}{\int_u^v e^{-ra} p_f(a)\varphi_f(a)da}} = \frac{1 - \dfrac{1}{\int_u^v e^{-r_0 a} p_f(a)\varphi_f(a)da}}{A(r_0)}$$

In the case of the human species, the quantities $A(r_0)$ are always close to A_0, and we therefore have, approximately:

$$\varepsilon = \frac{\int_u^v e^{-r_0 a} p_f(a)\varphi_f(a)da - 1}{A_0 \int_u^v e^{-r_0 a} p_f(a)\varphi_f(a)da}$$



Graph II.6. Graphic illustration of the second method of determining ρ by successive approximations

and if we put:

$$\int_u^v e^{-r_0 a} p_f(a) \varphi_f(a) da = 1 + \delta$$

we have:

$$\varepsilon = \frac{\delta}{A_0(1+\delta)}$$

If r_0 is sufficiently close to the value of ρ, however, the quantity δ is small, and in these circumstances:[15]

$$\varepsilon = \frac{\delta}{A_0} \tag{II.19}$$

A special case

There is one case in which the computation of the rate of variation ρ becomes very easy. This is where the fertility function $\varphi_f(a)$ amounts to a single value equal to the gross reproduction rate R′. Such a simplification of the fertility function would at first sight seem very artificial, since it is manifestly impossible for a woman to bring into the world at a given time R′ daughters. As will be seen later, however, stable populations computed with a simplified fertility function in this way are identical with stable populations computed with actually observed fertility functions, provided that the age to which the fertility function is reduced is close to the average age of mothers at the birth of their children. In such circumstances, the simplification of the fertility function is no more than a straightforward mathematical device.

The median age 27.5 years of the 25-29 five-year age group is close to the average age of mothers at the birth of their children. It is therefore particularly convenient to take this age and to consider the fertility function as being reduced to $\varphi_f(27.5) = R'$.

Equation (1) is written:

$$e^{-27.5r} p_f(27.5) \varphi_f(27.5) = 1$$

whence we have:

$$\varrho = \frac{\text{Log}\, p_f(27.5) + \text{Log}\, \varphi_f(27.5)}{27.5\, \text{Log}\, e}$$

or

$$\varrho = \frac{\text{Log}\, p_f(27.5) + \text{Log}\, \varphi_f(27.5)}{11.9431}$$

If, for example, we take a gross reproduction rate of 3.0

$$R' = \varphi_f(27.5) = 3.0$$

and use the intermediate model life table corresponding to an expectation of life at birth for both sexes of 40 years, we have the following calculation:

$p_f(27.5)$ = 0.62222 (taken from the model life table)[16]

Log 0.62222 = $\bar{1}.793{,}9440$

Log 3.0 = 0.477,1213

Log $p_f(27.5)$ + Log $\varphi_f(27.5)$ = 0.2710653

$$\varrho = \frac{0.2710653}{11.9431} = 0.022705$$

The age structure in the stable state

When we know the rate of variation in the stable state, it is easy to calculate the age structure by applying the method given in the first example.

Tables II.10 to II.13 give details of the calculations for the case considered here, where the gross reproduction rate R′ is 1.50. In the calculations, a value of ρ = 0.00087 has been used. Table II.13 gives details of the calculation of the age composition of the total population of both sexes together. We had omitted this calculation in the first example.

---

[15] We see here the formula proposed by Coale in his article "A new method for calculating Lotka's r, the intrinsic rate of growth in a stable population", *Population Studies*, vol. XI, No. 1, July 1957.

[16] *Methods of Estimating Population, Manual III: Methods for Population Projections by Sex and Age*, appendix, table III.

TABLE. II.10. COMPUTATION OF e^{-ra} AND e^{-ra} FOR $r = 0.0087$

Median age a	*Age group (years)*	ra	$ra \times \log e = \log e^{+ra}$	e^{+ra}	colog e^{ra}	e^{-ra}
2.5 . . .	0-4	0.0218	0.009468	1.0220	$\bar{1}.990532$	0.97844
7.5 . . .	5-9	0.0652	0.028316	1.0674	$\bar{1}.971684$	0.93688
12.5 . . .	10-14	0.1088	0.047251	1.1159	$\bar{1}.952749$	0.89691
17.5 . . .	15-19	0.1522	0.066100	1.1654	$\bar{1}.933890$	0.85880
22.5 . . .	20-24	0.1958	0.085035	1.2173	$\bar{1}.914965$	0.82218
27.5 . . .	25-29	0.2392	0.103883	1.2702	$\bar{1}.896117$	0.78726
32.5 . . .	30-34	0.2828	0.122818	1.3268	$\bar{1}.877182$	0.75367
37.5 . . .	35-39	0.3262	0.141667	1.3857	$\bar{1}.858333$	0.72166
42.5 . . .	40-44	0.3698	0.160602	1.4474	$\bar{1}.839398$	0.69087
47.5 . . .	45-49	0.4132	0.179450	1.5116	$\bar{1}.820540$	0.66152
52.5 . . .	50-54	0.4568	0.198386	1.5790	$\bar{1}.801614$	0.63331
57.5 . . .	55-59	0.5002	0.217234	1.6491	$\bar{1}.782766$	0.60641
62.5 . . .	60-64	0.5438	0.236169	1.7225	$\bar{1}.763831$	0.58084
67.5 . . .	65-69	0.5872	0.255018	1.7999	$\bar{1}.744982$	0.55588
72.5 . . .	70-74	0.6308	0.273953	1.8791	$\bar{1}.726047$	0.53227
77.5 . . .	75-79	0.6742	0.292801	1.9625	$\bar{1}.707199$	0.50956
82.5 . . .	80-84	0.7178	0.311737	2.0499	$\bar{1}.688263$	0.48782
87.5 . . .	85 +	0.7830	0.340053	2.1880	$\bar{1}.659947$	0.45703

TABLE II.11. COMPUTATION OF THE AGE DISTRIBUTION OF THE FEMALE STABLE POPULATION CORRESPONDING TO A RATE OF VARIATION OF $r = +0.0087$. INTERMEDIATE MODEL LIFE TABLE GIVING AN EXPECTATION OF LIFE AT BIRTH FOR BOTH SEXES OF 60.4 YEARS. INTERMEDIATE MODEL FEMALE FERTILITY GIVING A GROSS REPRODUCTION RATE OF 1.50

Median age a	*Age group (years)*	*Female stationary population (per 100,000 births)* L_a	e^{-ra}	*Female stable population (product of the two preceding columns)*	*Age distribution of the female stable population* C_a
2.5	0-4	460 386	0.97844	450 460	98 137
7.5	5-9	448 010	0.93688	419 732	91 444
12.5	10-14	444 150	0.89691	398 363	86 787
17.5	15-19	439 970	0.85880	377 846	82 318
22.5	20-24	434 040	0.82218	356 859	77 745
27.5	25-29	427 035	0.78726	336 188	73 242
32.5	30-34	419 610	0.75367	316 247	68 898
37.5	35-39	411 672	0.72166	297 087	64 723
42.5	40-44	402 742	0.69087	278 242	60 706
47.5	45-49	391 728	0.66152	259 136	56 455
52.5	50-54	377 275	0.63331	238 932	52 054
57.5	55-59	357 718	0.60641	216 924	47 259
62.5	60-64	330 472	0.58084	191 852	41 797
67.5	65-69	291 642	0.55588	162 118	35 319
72.5	70-74	238 028	0.53227	126 895	27 602
77.5	75-79	171 308	0.50956	87 292	19 017
82.5	80-84	102 020	0.48782	49 767	10 842
87.5	85 +	56 792	0.45703	25 956	5 655
ALL AGES . . .		6 204 598		4 589 696	1 000 000

The stable female birth rate is written:

$$b_f = \frac{100{,}000}{4{,}589{,}696} = 21.79 \text{ per thousand.}$$

The stable female death rate is written:

$$d_f = 21.79 - 8.70 = 13.09 \text{ per thousand.}$$

TABLE II.12. COMPUTATION OF THE AGE DISTRIBUTION OF THE MALE STABLE POPULATION CORRESPONDING TO A RATE OF VARIATION OF $r = +0.0087$. INTERMEDIATE MODEL LIFE TABLE GIVING AN EXPECTATION OF LIFE AT BIRTH FOR BOTH SEXES OF 60.4 YEARS. INTERMEDIATE MODEL FEMALE FERTILITY GIVING A GROSS REPRODUCTION RATE OF 1.50

Median age a	*Age group (years)*	*Male stationary population (per 100,000 births)* L_a	e^{-ra}	*Male stable population (product of the two preceding columns)*	*Age distribution of the male stable population* C_a
2.5	0-4	453 493	0.97844	443 716	101 037
7.5	5-9	440 230	0.93688	412 443	93 916
12.5	10-14	436 230	0.89691	391 259	89 092
17.5	15-19	431 860	0.85880	370 881	84 452
22.5	20-24	425 305	0.82218	349 677	79 624
27.5	25-29	417 588	0.78726	328 750	74 624
32.5	30-34	409 715	0.75367	308 790	70 314
37.5	35-39	401 195	0.72166	289 526	65 928
42.5	40-44	391 022	0.69087	270 536	61 603
47.5	45-49	377 658	0.66152	249 828	56 888
52.5	50-54	359 368	0.63331	227 591	51 824
57.5	55-59	334 412	0.60641	202 791	46 177
62.5	60-64	300 470	0.58054	174 435	39 720
67.5	65-69	255 450	0.55588	142 000	32 334
72.5	70-74	199 095	0.53227	105 972	24 131
77.5	75-79	135 710	0.50956	69 152	15 746
82.5	80-84	75 538	0.48782	36 849	8 391
87.5	85 +	38 093	0.45703	17 410	3 964
ALL AGES . . .		5 882 432		4 391 606	1 000 000

The stable male birth rate is written:

$$b_m = \frac{100{,}000}{4{,}391{,}600} = 22.77 \text{ per thousand.}$$

The stable male death rate is written:

$$d_m = 22.77 - 8.70 = 14.07 \text{ per thousand.}$$

TABLE II.13. COMPUTATION OF THE AGE DISTRIBUTION OF THE STABLE POPULATION (BOTH SEXES TOGETHER) CORRESPONDING TO A RATE OF VARIATION OF $r = +0.0087$. INTERMEDIATE MODEL LIFE TABLE GIVING AN EXPECTATION OF LIFE AT BIRTH FOR BOTH SEXES OF 60.4 YEARS. INTERMEDIATE MODEL FEMALE FERTILITY GIVING A GROSS REPRODUCTION RATE OF 1.50

Age group (years)	*Male stable population multiplied by 1.05*	*Female stable population*	*Stable population (both sexes)*	*Age distribution of the stable population (both sexes)*
0-4	465 902	450 460	916 362	99 595
5-9	433 065	419 732	852 797	92 686
10-14	410 822	398 363	809 185	87 946
15-19	389 425	377 946	767 271	83 391
20-24	367 161	356 859	724 020	78 690
25-29	345 188	336 188	681 376	74 055
30-34	324 230	316 247	640 477	69 610
35-39	304 002	297 087	601 089	65 329
40-44	284 063	278 242	562 305	61 157
45-49	262 319	259 136	521 455	56 674
50-54	238 971	238 932	477 903	51 941
55-59	212 931	216 924	429 855	46 719
60-64	183 157	191 852	375 009	40 758
65-69	149 100	162 118	311 218	33 825
70-74	111 271	126 695	237 966	25 863
75-79	72 610	87 292	159 902	17 379
80-84	38 691	49 767	88 458	9 614
85 + ...	18 280	25 956	44 236	4 808
ALL AGES	4 611 188	4 589 696	9 200 884	1 000 000

The stable birth rate for both sexes together is written:

$$b = \frac{205{,}000}{9{,}200{,}884} = 22.28 \text{ per thousand.}$$

The arithmetic mean of the stable female and male birth rates is:

$$\frac{21.79 + 22.77}{2} = 22.28 \text{ per thousand.}$$

The stable death rate for both sexes togegher is written: $d = 22.28 - 8.70 = 13.58$.
The arithmetic mean of the stable female and male death rates is:

$$\frac{13.09 + 14.07}{2} = 13.58 \text{ per thousand.}$$

Computation of the stable birth rate

The stable birth rate of the female population can be computed as in the first example

$$b_f = \frac{1}{\int_0^\infty e^{-\varrho a} p_f(a) da}$$

This is in fact a by-product of the preceding computation. The rates will be found at the foot of the tables.

We can establish approximate formulae for the computation of the stable birth rate, just as we did for the computation of the intrinsic rate of natural variation ρ. These formulae involve not only the moments and cumulants of the function $p_f(a)\varphi_f(a)$, but also the moments and cumulants of the survivorship function $p_f(a)$.

First approximation

$$b_f = \frac{R_0}{E_0} \qquad \text{(II.20)}$$

where R_0 is the net reproduction rate and E_0 is the first-order moment of the function $p_f(a)$, i.e., the expectation of life at birth E_0.

Second approximation

$$b_f = \frac{R_0}{E_0} \cdot \frac{1 - \varrho\mu_1}{1 - \varrho\lambda_1} \qquad \text{(II.21)}$$

where μ_1 is the first-order cumulant of the function $p_f(a)\varphi_f(a)$, λ_1 is the first-order cumulant of the function $p_f(a)$, and ρ is the intrinsic rate of variation.

The same formulae can be used to compute the stable male birth rate, provided that the male survivorship function is used instead of the female survivorship function in calculating the expectation of life and the cumulant λ_1.

Tables II.14 and II.15 give details of the application of these formulae. We know that:

$$\lambda_1 = \frac{E_1}{E_0}$$

where

$$E_0 = \int_0^\infty p(a) da$$

and

$$E_1 = \int_0^\infty a p(a) da$$

Consequently, λ_1 is simply the mean age of the stationary population (table II.14). μ_1 has already been determined (table II.8). Thus, we have the calculations given in table II.15.

When the intrinsic rate of variation is *positive*, it can be demonstrated that the first and second approximations encircle the true value. We thus have an idea of the error resulting from use of the approximate formula. If the rate of variation is negative, however, we have no idea what this error will be.

TABLE II.14. COMPUTATION OF THE MEAN AGE OF THE FEMALE STATIONARY POPULATION. INTERMEDIATE MODEL LIFE TABLE GIVING AN EXPECTATION OF LIFE AT BIRTH FOR BOTH SEXES OF 60.4 YEARS. INTERMEDIATE MODEL FEMALE FERTILITY GIVING A GROSS REPRODUCTION RATE OF 1.50

Median age a	*Age group (years)*	*Female stationary population per 100 000 births*	*Product of the preceding column and the median age*
2.5	0-4	460 386	1 150 960
7.5	5-9	448 010	3 360 080
12.5	10-14	444 150	5 551 880
17.5	15-19	439 970	7 699 480
22.5	20-24	434 040	9 765 900
27.5	25-29	427 035	11 743 460
32.5	30-34	419 610	13 637 320
37.5	35-39	411 672	15 437 700
42.5	40-44	402 742	17 116 540
47.5	45-49	391 728	18 607 080
52.5	50-54	377 275	19 806 940
57.5	55-59	357 718	20 568 780
62.5	60-64	330 472	20 654 500
67.5	65-69	291 642	19 685 840
72.5	70-74	238 028	17 257 030
77.5	75-79	171 308	13 276 370
82.5	80-84	102 020	8 416 650
87.5	85 +	56 792	5 111 280
	ALL AGES	6 204 598	228 847 790

$$\lambda_1 = \frac{228{,}847{,}790}{6{,}204{,}598} = 36.884 \text{ years.}$$

Age composition of deaths

The method given in the first example is used. We have not reproduced the calculation here.

THIRD EXAMPLE: THE CRUDE BIRTH RATE b_0 IS KNOWN

In this case, we can write the r equation:

$$\int_0^{\omega} e^{-ra} p(a)da = \frac{1}{b} \qquad \text{(II.1)}$$

In order to solve this equation, let us consider the integral:

$$\mathrm{J}(r) = \int_0^{\omega} e^{-ra} p(a)da$$

J (r) a decreasing function of r. For infinitely great values of r it takes the following values:[17]

When r tends towards $+\infty$, J(r) tends towards zero; and when r tends towards $-\infty$, J(r) tends towards $+\infty$.

The curve representing b_r is thus of the form shown in graph II.7. The straight line of the ordinate b_0 cuts this curve at a single point M. Equation (II.1) has thus only

TABLE II.15. COMPUTATION OF THE STABLE FEMALE BIRTH RATE BY APPROXIMATION

A. First formula: $b_f = \frac{\mathrm{R}_0}{\mathrm{E}_0}$

Gross reproduction rate	*Zero-order moment of the function:* $p_f(a)\varphi_f(a)$	*Female expectation of life at birth:* E_0	*Stable female birth rate,* b_f *(per thousand)*
1.50	1.276950	62.04598	20.58
0.75	0.638475	62.04598	10.29

B. Second formula: $b_f = \frac{\mathrm{R}_0}{\mathrm{E}_0} \cdot \frac{1 - \rho\mu_1}{1 - \rho\Upsilon_1}$

Gross reproduction rate	ϱ	μ_1	λ_1	$\varrho\mu_1$	$\varrho\lambda_1$	$1-\varrho\mu_1$	$1-\varrho\lambda_1$	$\frac{1-\varrho\mu_1}{1-\varrho\lambda_1}$	bf
1.50	+0.0087	27.9521	36.884	0.24318	0.32089	0.75682	0.67911	1.114	22.93
0.75	−0.0157	27.9521	36.884	−0.43885	−0.57908	1.43885	1.57908	0.911	9.37

C. Comparison of various values of the stable female birth rate (per 1,000)

Gross reproduction rate	*First approximate formula*	*Second approximate formula*	*Rate computed by numerical integration*
1.50	20.58	22.93	21.79
0.75	10.29	9.37	8.46

---

[17] As already pointed out, all that is needed in order to see how J(r) behaves for infinitely great values of r is to imagine J(r) written in discontinuous notation.

(*a*) When r tends towards $+\infty$, the first term (that corresponding to the lowest age) becomes preponderant over all the others, and consequently J(r) behaves like $e^{-r\varepsilon}\mathrm{L}_\varepsilon$, where ε is a small but fixed quantity. When r tends towards $+e^{-r\varepsilon}$, L_ε tends towards zero.

(*b*) When r tends towards $-\infty$, the last term becomes preponderant over all the others, and J(r) behaves like $e^{-r(w-\varepsilon)}\mathrm{L}_{w-\varepsilon}$: a quantity which tends towards $+\infty$.

one real root r_0, whose value is the abscissa of the point M. In order to calculate r_0 in practice, we can use a graphic method, of which graph II.7 gives the principles and graph II.8 gives a specific illustration.

In graph II.8 we have taken the same sub-set $H_0(r)$ as in the second example, i.e., we have taken the sub-set of all the Malthusian populations corresponding to a level-80 of intermediate model life table, with an expectation of life at birth for both sexes of 60.4 years. We have seen that the stable population of $H_0(r)$ corresponding to a gross reproduction rate[18] of $R' = 1.50$ has as its intrinsic rate of natural variation $\rho = 0.0087$ and as its stable birth rate for both sexes together $b = 0.02228$ (table II.13).

Let us now determine which population of the sub-set $H_0(r)$ has a birth rate (both sexes) of $b_0 = 0.02228$, i.e., exactly equal to the birth rate (both sexes) of the stable population whose characteristics we have just given.

In graph II.8 we have drawn, for both sexes together, the curve showing the variation of

$$\frac{1}{J(r)}$$

in the sub-set $H_0(r)$ under consideration.[19] The straight line of $b = 22.28$ per thousand cuts this curve at a point M whose abscissa r must equal the value of the intrinsic rate of natural variation of the population which we have just given (i.e., $\rho = 0.0087$).

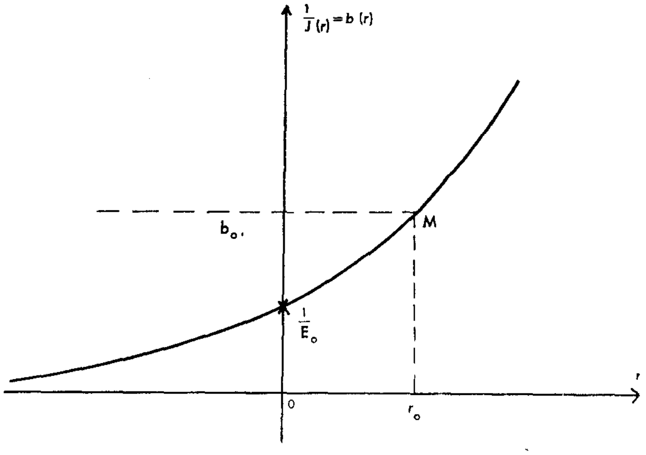
On the scale of the graph, we can indeed see that r_0 is between 0.008 and 0.009. All that is necessary in order to determine r_0 more precisely is to draw a graph on a larger scale. We can also, however, use a method by successive approximations similar to that described in the second example.

If r_1 is an approximate value of r_0, the true value of which is $r_0 = r_1 + \varepsilon$, we can write:

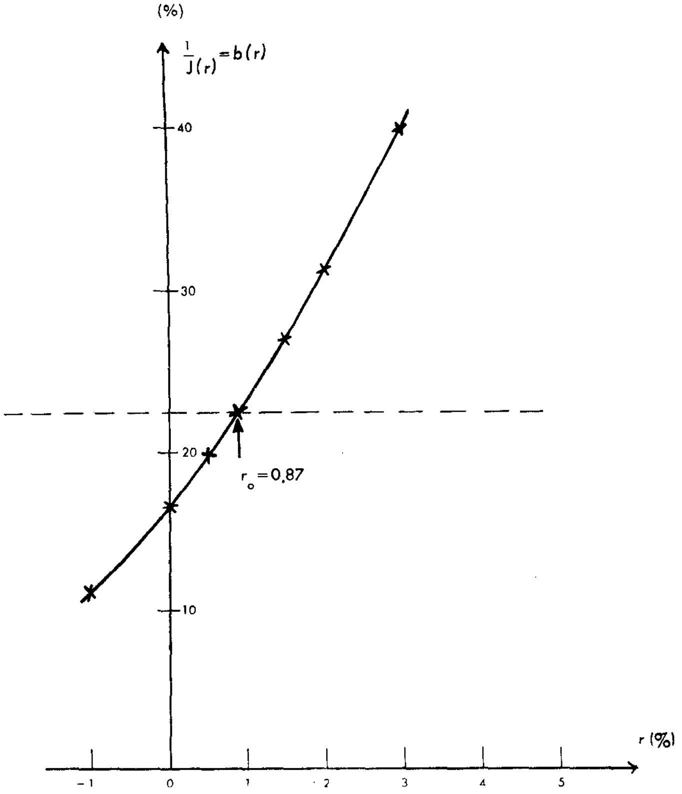
$$\int_0^\omega e^{-(r_1+\varepsilon)A} p(a)da = \frac{1}{b}$$

and as ε is small, if we assimilate $e^{-\varepsilon a}$ to $1 - \varepsilon a$ we find that:

$$\varepsilon = \frac{\int_0^\omega e^{-r_1 a} p(a)da - \frac{1}{b}}{\int_0^\omega a e^{-r_1 a} p(a)da} \qquad \text{(II.22)}$$



Graphic II.7. Form of the curve representing $\frac{1}{J(r)} = b\ (r)$ as a function of r in a sub-set $H_0(r)$ of Malthusian populations, i.e., a sub-set of all Malthusian populations belonging to the same life table



Graph II.8. Graphic determination of the rate of increase r_0 of the Malthusian population of the sub-set $H_0(r)$ associated with the intermediate model life table giving an expectation of life at birth for both sexes of 60.4 years and based on a crude birth rate (boys and girls) of 22.28 per thousand

In the particular case considered here, if we take $r_1 = 0.008$ we obtain $\varepsilon = 0.0007$, which does indeed give $r_0 = 0.0087$. Table II.16 gives details of the computation as it is performed for the female sex.

FOURTH EXAMPLE: THE CRUDE DEATH RATE d_0 IS KNOWN

In this case we can write the r equation:

$$b - r = \frac{1}{\int_0^\omega e^{-ra} p(a)da} - r = d_0 \qquad \text{(II.2)}$$

which is simply equation (II.2) in table II.2. In order to solve this equation, let us consider the equation:

$$K(r) = \frac{1}{\int_0^\omega e^{-ra} p(a)da} - r$$

When r tends towards $\pm\infty$, this equation tends towards $+\infty$.[20]

---

[18] The age distribution of female fertility rates being those of the intermediate model.

[19] We can plot the variation curves in this way for either the female or the male sex. The same applies to all the examples which follow.

[20] When r approaches $+\infty$, K(r) behaves like $e^{-r\varepsilon}/L_\varepsilon - r$ where ε is small but fixed. In these conditions, $e^{r\varepsilon}$ preponderates over r and consequently K(r) approaches $+\infty$. When r approaches $-\infty$, K(r) behaves like $(e^{r(\omega-\varepsilon)}/L_{\omega-\varepsilon}) - r$. As r is negative and infinitely great, $e^{r(\omega-\varepsilon)}/L_\varepsilon$ approaches zero and $-r$ approaches $+\infty$. K(r) thus approaches $+\infty$.

TABLE II.16. APPLICATION OF THE METHOD OF SUCCESSIVE APPROXIMATIONS IN ORDER TO COMPUTE THE RATE OF A MALTHUSIAN POPULATION WHOSE MORTALITY AND CRUDE BIRTH RATE ARE KNOWN (FORMULA II.22 IN THE TEXT). LEVEL-80 FEMALE MODEL MORTALITY RATE (EXPECTATION OF LIFE AT BIRTH FOR BOTH SEXES OF 60.4 YEARS). FEMALE CRUDE BIRTH RATE $b_0 = 0.02179$ (VALUE TAKEN FROM TABLE II.11). APPROXIMATE VALUE OF r_0, $r_1 = 0.008$

Median age of age group a *(1)*	*Age group (years) (2)*	$r\ a$ *for* $r = 0.008$ *(3)*	ra Log e *(4)*	e^{ra} *(5)*	*Stationary population* L_a *(6)*	*Quotient of (6) divided by (5)* $e^{-ra}L_a$ *(7)*	*Preceding column multiplied by* a *(8)*
2.5	0-4	0.02	0.00869	1.020	460 386	451 390	1.1285
7.5	5-9	0.06	0.02606	1.062	448 010	421 900	3.1695
12.5	10-14	0.10	0.04343	1.105	444 150	401 950	5.0250
17.5	15-19	0.14	0.06080	1.150	439 970	382 610	6.6970
22.5	20-24	0.18	0.07817	1.197	434 040	362 610	8.1600
27.5	25-29	0.24	0.09554	1.246	427 035	342 770	9.4260
32.5	30-34	0.26	0.11292	1.297	419 610	323 600	10.5850
37.5	35-39	0.30	0.13029	1.350	411 672	304 950	11.4370
42.5	40-44	0.34	0.14766	1.405	402 742	286 690	12.1850
47.5	45-49	0.38	0.16503	1.462	391 728	267 970	12.7360
52.5	50-54	0.42	0.18240	1.522	377 275	247 870	13.0170
57.5	55-59	0.46	0.19878	1.584	357 718	225 860	12.989
62.5	60-64	0.50	0.21715	1.649	330 472	200 160	12.511
67.5	65-69	0.54	0.23452	1.716	291 642	169 950	11.472
72.5	70-74	0.58	0.25189	1.786	238 028	133 220	9.658
77.5	75-79	0.62	0.26926	1.859	171 308	92 220	7.1420
82.5	80-84	0.66	0.28663	1.935	102 020	52 720	4.3490
87.5	85 +	0.70	0.30401	2.014	56 792	28 200	2.4680
ALL AGES						4 696 580	154.4155

$b_0 = 0.02179$ (value taken from table II.11).

$$\frac{1}{b_0} = 45.89. \qquad \varepsilon = \frac{46.97 - 45.89}{1\,544.155} = 0.0006978.$$

$$r_0 = r_1 + \varepsilon = 0.008 + 0.0006978 = 0.0086978.$$

Moreover:

$$\frac{d\mathrm{K}(r)}{dr} = \frac{\int_0^\omega ae^{-ra}p(a)da}{\left[\int_0^\omega e^{-ra}p(a)da\right]^2} - 1$$

We can easily see that the expression:

$$\frac{\int_0^\omega ae^{-ra}p(a)da}{\left[\int_0^\omega e^{-ra}p(a)da\right]^2}$$

increases from zero to $+\infty$ when r varies from $-\infty$ to $+\infty$. It thus takes once the value 1 and the derivative is set at zero for a value r_m of r. The curve representing $\mathrm{K}(r)$ is thus of the form shown in graph II.9. It passes through a minimum for $r = r_m$.

The straight line of ordinate d_0 cuts this curve at zero point or at one point or at two points. Let us take the case where there are two points of intersection M and M′; the abscissae r_0 and r'_0 of these two points are the real solutions of equation II.2. Consequently, if we assume a value for the crude death rate, there will be zero, one or two populations in the set $\mathrm{H}_0(r)$ with that death rate. The result depends on the value of d_0.

When there is only one solution, we have $r_0 = r'_0 = r_m$. The corresponding Malthusian population possesses a remarkable property, for we have for this population:

$$\frac{d\mathrm{K}(r)}{dr} = 0$$

which is written:

$$\frac{\int_0^\omega ae^{-ra}p(a)da}{\int_0^\omega e^{-ra}p(a)da} \times \frac{1}{\int_0^\omega e^{-ra}p(a)da} = 1$$

or $\qquad \mathrm{A}(r_m)b(r_m) = 1$

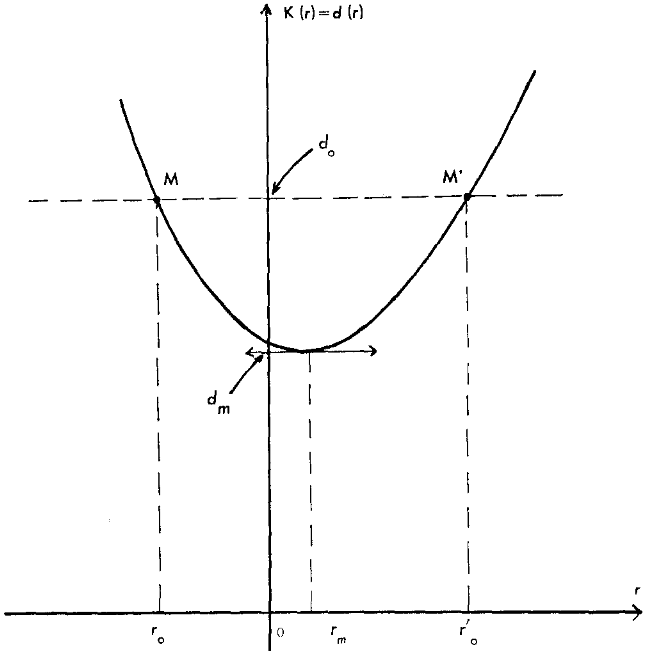
where $\mathrm{A}(r_m)$ is the mean age of the population. In other words, in the Malthusian population corresponding to the minimum of $\mathrm{K}(r)$, the mean age equals the inverse of the crude birth rate.

Finally, for a sub-set $\mathrm{H}_0(r)$, $\mathrm{K}(r)$ represents all the death rates of the Malthusian populations of that sub-set. We see that these death rates pass through a minimum d_m.

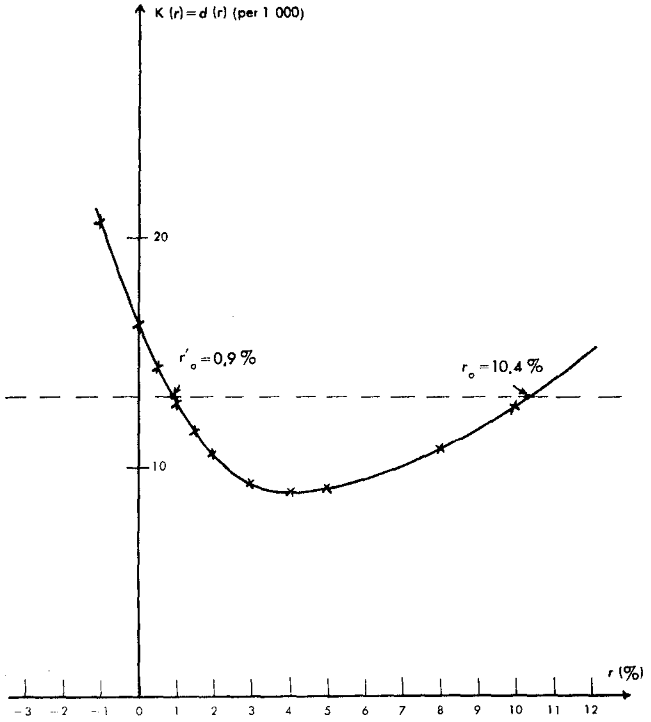
Graph II.9 gives the principle of the graphic solution of equation II.2, while graph II.10 gives a specific application of it. We take the same sub-set $\mathrm{H}_0(r)$ as in the two preceding examples (i.e., intermediate model life table with an expectation of life at birth for both sexes of 60.4 years). *For the female population,* the curve of graph II.10 is the curve of variation of $\mathrm{K}(r)$ in this sub-set.

In the second example considered above, the stable population corresponding to the gross reproduction rate $\mathrm{R}' = 1.50$ had as its crude female death rate $d_0 = 0.01309$, while the intrinsic rate of natural variation was $r_0 = 0.0087$.

Let us determine the Malthusian populations of the sub-set $\mathrm{H}_0(r)$ which have in fact crude female death rates of exactly the value obtained for the stable populations whose characteristics were referred to above. The straight line of the ordinate 0.01309 cuts the curve of graph II.10 at two points, M and M′, whose abscissae are the values of r which are sought. One of these values must be equal to the intrinsic rate of variation of the stable population which we have already computed, and this is in fact exactly what we obtain.



Graph II.9. Form of the curve representing the variation of K(r) = d(r) as a function of r in a sub-set H$_0$(r) of Malthusian populations, i.e., the sub-set of all Malthusian populations having the same life table



Graph II.10. Graphic determination of the rates of increase r_0 and r'_0 of the two Malthusian populations of the sub-set H$_0$(r), associated with the intermediate model life table giving an expectation of life at birth for both sexes of 60.4 years, for which the crude female death rate is 13.09 per thousand

At the scale of the graph we have:

$r_0 = 0.0090$

$r'_0 = 0.104$

If greater precision is desired, a graph on a larger scale must be traced. A method by successive approximation can also be used, as in the preceding examples. If r_1 is an approximate value of r_0 and if we assume that $r_0 = r_1 + \varepsilon$, we easily find by transferring this value of r_0 into II.2 and assimilating $e^{-\varepsilon a}$ to $1 - \varepsilon a$ that:

$$\varepsilon = \frac{1 - (d_0 + r_1)\int_0^{\omega} e^{-r_1 a}\, p(a)da}{\int_0^{\omega} e^{-r_1 a}\, p(a)da - (d_0 + r_1)\int_0^{\omega} a e^{-r_1 a} p(a)da}$$

This can also be written:

$$\varepsilon = \frac{b(r_1) - (d_0 + r_1)}{1 - (d_0 + r_1)\mathrm{A}(r_1)} = \frac{d(r_1) - d_0}{1 - (d_0 + r_1)\mathrm{A}(r_1)} \qquad \text{(II.23)}$$

where $d(r_1)$ is the death rate of the Malthusian population corresponding to r_1 and A(r_1) is the mean age of the same population.

Let us apply this formula for $r_1 = 0.008$. According to table II.16, we have:

$$b(r_1) = \frac{100\,000}{4\,696\,580} = 0.021292$$

and consequently $d(r_1) = 0.021292 - 0.008 = 0.013292$.

We also have, again according to table II.16:

$$\mathrm{A}(r_1) = \frac{154\,415\,500}{4\,696\,580} = 32.882$$

We therefore have:

$$\varepsilon = \frac{0.013292 - 0.013090}{1 - (0.013090 + 0.008000)32.882} = \frac{0.000202}{1 - 0.69355}$$

and finally:

$\varepsilon = 0.000659$, whence $r_0 = 0.008 + 0.000659 = 0.008659$,

i.e., roughly 0.0087. We thus arrive, once again, at the intrinsic rate of variation of the stable population.

As regards $r'_1 = 0.103$, there would obviously be another correction.

FIFTH EXAMPLE: THE AGE DISTRIBUTION $c(a_0)$ AT A GIVEN AGE a_0 IS KNOWN

In this case we must write the r equation:

$$\frac{e^{-ra_0}p(a_0)}{\int_0^{\omega} e^{-ra}p(a)da} = c(a_0) \qquad \text{(II.3}b\text{)}$$

This is formula (II.3b) in table II.2 Let us consider the equation:

$$\mathrm{U}(r) = \frac{e^{-ra_0}p(a_0)}{\int_0^{\omega} e^{-ra}p(a)da}$$

When r tends towards $\pm\infty$, $U(r)$ tends towards zero.[21] In addition, its derivative may be written:

$$\frac{dU(r)}{dr} =$$

$$\frac{-ae^{-ra_0}p(a_0)\int_0^{\omega} e^{-ra}p(a)da + e^{-ra_0}p(a_0)\int_0^{\omega} ae^{-ra}p(a)da}{\left[\int_0^{\omega} e^{-ra}p(a)da\right]^2}$$

which becomes:

$$\frac{dU(r)}{dr} = b(r)e^{-ra_0}p(a_0)\,[A(r) - a_0]$$

where $b(r)$ and $A(r)$ are, respectively, the crude birth rate and the mean age of the Malthusian population having a rate of increase r. As r increases from $-\infty$ to $+\infty$, the mean age of the population decreases from ω to zero. It assumes once the value of a_0, and for the corresponding value of r, i.e., r_m, the derivative is cancelled out.

Thus, the curve representing the variation of $U(r)$ is of the form shown in graph II.11 and passes through a maximum.[22]

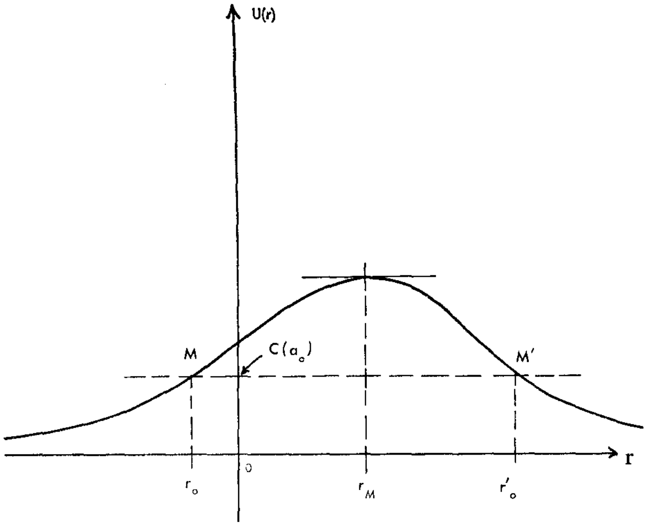
The straight line of ordinate $c(a_0)$ cuts this curve at zero point or at one point or at two points, according to the value of $c(a_0)$. If we take the case where there are two points of intersection, M and M′, then the abscissae r_0 and r'_0, of these two points are the real roots of equation II.3*b*. We thus have a result similar to that obtained in the fourth example.

Graph II.11 shows the principle of the graphic solution of equation II.3*b*, while graph II.12 gives a specific illustration of this. We continue to take the same sub-set $H_0(r)$ (intermediate model life table giving an expectation of life at birth for both sexes of 60.4 years). The curve of the graph for $U_0(r)$ is the curve representing the variation, in the sub-set in question, of the proportion C_{25-29} *of the female population.* We propose to determine, for the sub-set $H_0(r)$, the populations for which $C_{25-29}=0.073242$. This value of C_{25-29} is that of the stable population corresponding in the second example of $R' = 1.50$. In these conditions, one of the two values r_0 or r'_0 must be equal to the intrinsic rate of natural variation of this stable population, $\rho = 0.0087$. This is in fact what we observe in graph II.12. At the scale of this graph, we have the following approximate values:

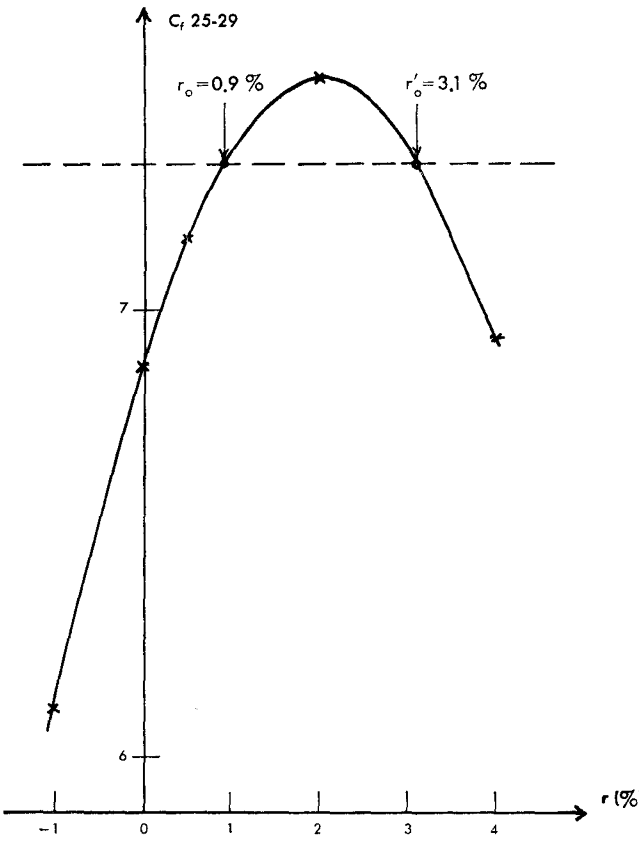
$r_0 = 0.0090$

$r'_0 = 0.0310$

As in the preceding examples, it is easy to apply a more accurate method of computation by successive approximations. No details are given here of these computations, since they are similar in all respects to the computations described in preceding examples.



Graph II.11. Form of the curve representing the variation of the integral U(r) as a function of r



Graph II.12. Graphic illustration of the rates of increase r_0 and r'_0 of the two Malthusian populations of the sub-set $H_0(r)$, associated with the intermediate model life table corresponding to an expectation of life at birth for both sexes of 60.4 years, in which women aged 25 to 29 represent 7.3 per cent of the total

SIXTH EXAMPLE: THE AGE DISTRIBUTION OF DEATHS $d(a_0)$ AT A GIVEN AGE a_0 IS KNOWN

In this case, we have to write the r equation:

$$d(a_0) = \frac{e^{-ra_0}p(a_0)q(a_0)}{\int_0^{\omega} e^{-ra}p(a)q(a)da} \qquad \text{(II.8)}$$

---

[21] When r approaches $+\infty$, $U(r)$ behaves like

$$\frac{e^{-ra0}p(a_0)}{e^{-r\varepsilon}}$$

The value ε can always be selected smaller than a_0, so that e^{-ra_0} will predominate over $e^{-r\varepsilon}$ and $U(r)$ will approach zero. When r tends towards $-\infty$, $U(r)$ behaves like

$$\frac{e^{-ra0}p(a_0)}{e^{-r(\omega-\varepsilon)}L_{\omega-\varepsilon}}$$

If the quantity ε is selected small enough, $e^{-r(\omega-\varepsilon)}$ predominates over e^{-ra_0}, since r is negative, and consequently $U(r)$ tends towards zero.

[22] At the maximum, the derivative is cancelled out and we therefore have $A(r) = a_0$. The mean age of the population is equal to the age a_0, for which we know the age structure $C(a_0)$.

This is formula II.8 in table II.2, and it is of the same form as formula II.3*b* in the fifth example. Let us consider the equation:

$$M(r) = \frac{e^{-ra_0}p(a_0)q(a_0)}{\int_0^{\omega} e^{-ra}p(a)q(a)da}$$

The curve representing M(r) as a function of r is of the same form as the curve representing U(r) (graph II.11). M(r) passes through a maximum, and the straight line of the ordinate $d(a_0)$ cuts it at zero point or at one point or at two points. The r equation (II.8) thus has no solution, one solution or two solutions, according to the value of $d(a_0)$.

Graph II.13 gives an example of the graphic solution of equation II.8. In this case, we have taken a sub-set $H_0(r)$ which is different from the sub-set considered in examples 2 to 5, being the one used in the first example,[23] associated with the level-60 of the intermediate model life table giving an expectation of life at birth for both sexes of 50 years. It will be recalled that in the first example we calculated the age distribution of female deaths in the Malthusian population of the sub-set $H_0(r)$ which had a rate of natural variation of 0.03, and we found that 22.82 per cent of female deaths occurred in the 40-44 age group.

Let us now determine the Malthusian populations of the sub-set $H_0(r)$ in which the proportion of female deaths in the 40-44 age group is exactly 22.82 per cent. We must find the Malthusian population whose rate of variation is 0.03, but there is also another one, as can be seen from graph II.13, on which we have plotted the curve representing the variation of M(r) as a function of r for the 40-44 age group. The straight line of the ordinate of 0.2282 cuts this curve at two points M and M′ whose abscissae are:

$$r_0 = -0.0125$$

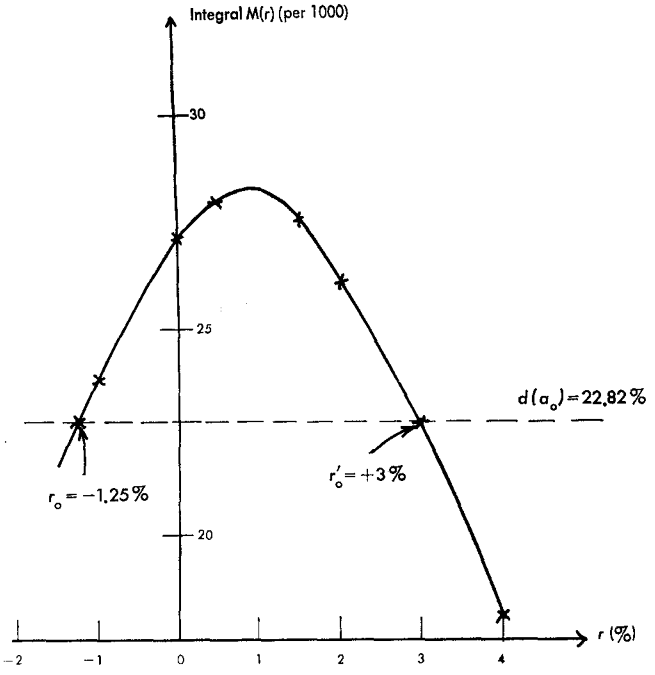
and

$$r'_0 = +0.030$$

Thus we find, once again, the exponential population with a rate of variation of 0.03, and another with a rate of variation of −0.0125.

Obviously, the graphic solution of equation II.8 is an imprecise method. Successive approximations enable us to calculate more accurate values similar to that worked out in the previous examples.

We shall confine ourselves to these six examples, but many others can readily be imagined. Instead of assuming the age distribution at a given age, for example, we can assume the ratio between two age groups, such as the ratio of the number of old people to the number of persons of working age, or any other combination of ages. All these problems are treated in the same way. From the mathematical point of view, they all lead to an equation in r, and all that is necessary is to find the real roots of that equation.



Graph II.13. Graphic determination of the rates of increase r_0 and r'_0 of the two Malthusian populations of the sub-set H(r) associated with the intermediate model life table giving an expectation of life at birth for both sexes of 50 years, in which the proportion of deaths of women aged 40 to 44 represents 22.82 per cent of total female deaths

In all these examples, the value or values of r used in the equation are exact values. If we use approximate values and content ourselves with solving equations whose solutions are "statistical" variables diverging to a greater or less extent from an average value, we begin to encounter other problems, of which we shall now give two examples.

COMPATIBILITY OF A GIVEN AGE DISTRIBUTION OF THE POPULATION WITH A GIVEN MORTALITY

For a given sub-set $H_0(r)$, we cannot make an arbitrary assumption of the whole of the age distribution $C_0(a)$. Indeed, we showed in the fifth example that we need only know the age distribution at a single age $C_0(a_0)$ in order to know the age distribution for all ages.[24]

Let us, then, consider an actual population, and let $p_0(a)$ and $c_0(a)$ be the survivorship function and the age distribution observed at a given time. Let us consider the sub-set $H_0(r)$ corresponding to $p_0(a)$, and let us pose the following question: does the age distribution $C_0(a)$ belong to the sub-set $H_0(r)$?

If we want $C_0(a)$ to coincide exactly with the age distribution of a population of the sub-set $H_0(r)$, we shall generally have to answer this question in the negative. It will be recalled from what was stated above in connexion with the fifth example that each term of the sequence $C_0(a)$ determines a particular population of the sub-set $H_0(r)$, and it is unlikely that all the particular populations thus determined will be *absolutely* identical. It may happen that all the populations are *almost* identical, and if we are willing to accept an approximate coincidence, the question which we have posed takes on a totally different meaning.

---

[23] The first example is the only one in which we calculated the age distribution of deaths in a Malthusian population. That is why we have reverted to this example to illustrate the case where the age distribution of deaths at a given age a_0 is known.

[24] Obviously, we are assuming the case where the straight line of ordinate $C_0(a_0)$ intersects the curve of graph II.11.

If the populations coincide exactly, then there are two numbers b_0 and r_0 which are such that:

$$b_0 e^{-r_0 a} p_0(a) = C_0(a)$$

If we take the logarithms, this is written:

$$\text{Log } b_0 - r_0 a \text{ Log } e = \text{Log } C_0(a) - \text{Log } p_0(a) \qquad \text{(II.23)}$$

Let us now assume that:

$$y = \text{Log } p_0(a) - \text{Log } C_0(a)$$

and

$$x = a,$$

we shall then have:

$$y = x r_0 \text{ Log } e - \text{Log } b_0 \qquad \text{(II.24)}$$

To each value of a there corresponds a pair of values (x, y), and if we plot a graph with x on the horizontal axis and y on the vertical axis the points obtained will be on a straight line given by the equation II.24.

If there is only an approximate coincidence, we shall obtain a cluster of points which can be more or less adjusted by the straight line of equation II.24.

Let us see how the problem occurs in a specific example. Table II.17 gives the age distribution of the female population of Mexico according to the 1940 census and the life table computed for the same population for 1940. In order to apply the foregoing formulae in discontinuous form, we must use the approximate formulae described earlier:

$$5C_0(22.5) = C_{20\text{-}24}$$

and

$$5p_0(22.5) = L_{20\text{-}24}$$

We then see that:

$$y = \text{Log } p_0(22.5) - \text{Log } C_0(22.5) = \\ = \text{Log } L_{20\text{-}24} - \text{Log } C_{20\text{-}24}$$

and we shall assume that $x = 22.5$

Table II.17 gives details of the calculation of y for successive age groups, and graph II.14 shows that the set of points (x, y) can be adjusted by a straight line. It is true that there are some points which diverge from the adjustment line fitted, particularly at the two extremities, but it is well known that young children (0 and 1-4 years of age) and old people (80 and over) are often incorrectly recorded in censuses, and it is therefore to be expected that the points corresponding to very young and very old people should show some divergences.

The ordinate at the origin of the adjustment line is written according to formula II.23: $-\text{Log } b_0 = 1.29$, whence $\text{Log } b_0 = 2.71$, which gives us: $b_0 = 51.29$ per thousand.

TABLE II.17. COMPUTATION OF THE QUANTITIES y OF FORMULA (II.24) IN THE TEXT, WITH A VIEW TO STUDYING THE COMPATIBILITY OF THE AGE DISTRIBUTION OF THE FEMALE POPULATION OF MEXICO AS DETERMINED IN THE 1940 CENSUS, C_a, WITH THE LIFE TABLE COMPUTED FOR THE SAME POPULATION AT THE SAME DATE, L_a

Median age, $a = x$	*Age group (years)*	*Female population according to 1940 census,* C_a	*Life table for Mexico in 1940,* L_a	Log C_a	Log L_a	*Difference* Log. L_a — Log C_a $= y$
0.5	0	2 628	88 740 (b)	3.41963	4.94812	1.52849
2.5	1-4	11 599	311 000 (c)	4.06446	5.49276	1.42830
7.5	5-9	13 936	349 092 (d)	4.14457	5.54294	1.39837
12.5	10-14	11 611	338 925	4.06483	5.53011	1.46528
17.5	15-19	10 314	330 632	4.01326	5.51934	1.50608
22.5	20-24	8 114 (a)	318 650	3.90924	5.50331	1.59407
27.5	25-29	8 432 (a)	304 600	3.92593	5.48373	1.55780
32.5	30-34	6 874	290 035	3.83721	5.46244	1.62523
37.5	35-39	7 041	274 487	3.84763	5.43853	1.59090
42.5	40-44	4 897	257 710	3.68993	5.41113	1.72120
47.5	45-49	3 970	239 580	3.59879	5.37945	1.78066
52.5	50-54	3 182	219 947	3.50270	5.34232	1.83962
57.5	55-59	2 205	197 375	3.34341	5.29530	1.95189
62.5	60-64	2 157	168 947	3.33385	5.22776	1.89391
67.5	65-69	1 158	134 345	3.06371	5.12824	2.06453
72.5	70-74	845	96 272	2.92686	4.98350	2.05664
77.5	75-79	450	59 575	2.65321	4.77506	2.12185
82.5	80-84	336	30 702	2.52634	4.48717	1.96083
87.5	85 +	251	20 180 (e)	2.39967	4.30492	1.90535
ALL AGES . . .		100 000				

SOURCE: For the age distribution C_a, see *Demographic Yearbook, 1940* (United Nations publication, Sales No.: 49.XIII.1). For the life table, see *Demographic Yearbook, 1961* (United Nations publication, Sales No.: 62.XIII.1).

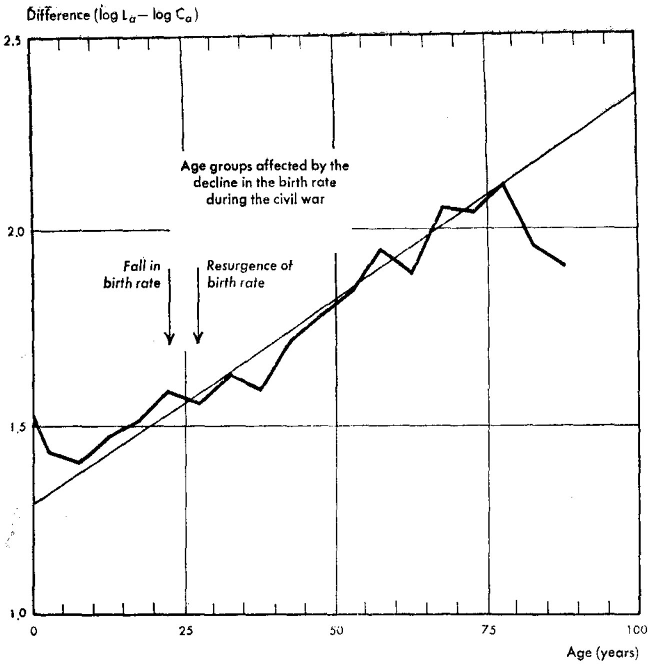
(a) Decline in birth rate during the revolution of 1911-1921, followed by a resurgence of the birth rate after the civil war. If these events were ignored it would be necessary to correct the age distribution. A graphic adjustment would involve taking 8,814 instead of 8,114 for the 20-24 age group and 7,732 instead of 8,432 for the 25-29 age group.

(b) $L_0 = 0.25p(0) + 0.75p(1)$.

(c) $L_1 = 1.9p(1) + 2.1p(5)$.

(d) $L_{5\text{-}9} = \frac{p(5) + p(10)}{2} \times 5$. This formula is valid for all five-year age groups.

(e) $L_{85+} = p(85) \times \text{Log } p(85)$, $p(85)$ being expressed per 100,000 births.



Graph II.14. Compatibility of the age distribution of the female population of Mexico, as determined in the 1940 census, with the life table computed for the same population at the same date

The ordinate of the point corresponding to 100 years on the horizontal axis equals:

$$\text{Log } b_0 + 100\, r_0 \text{ Log } e = 2.37$$

We therefore have: $100\, r_0 \text{ Log } e = 1.08$, whence [25]

$$r_0 = \frac{1.08}{43.429} = 0.02487$$

Finally, the crude death rate is:

$$d_0 = b_0 - r_0 = 26.42 \text{ per thousand}$$

Thus, the female population of Mexico, as recorded in the 1940 census, coincides quite well with a Malthusian population whose death rate is given in the life table in table 11.17 and whose crude birth and death rates and rate of increase are, respectively:

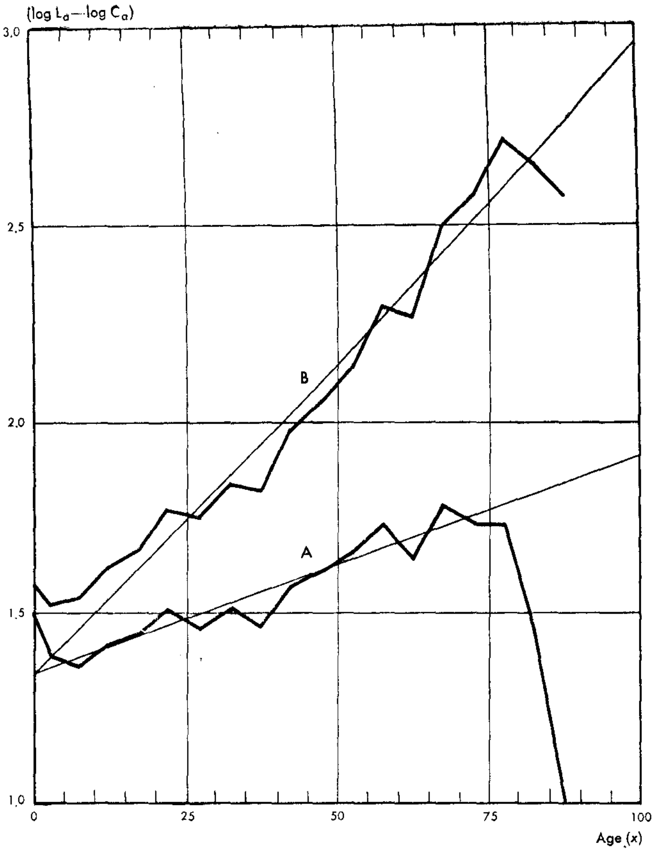
$b_0 = 51.29$ (per thousand)

$d = 26.42$ (per thousand)

$r_0 = 24.87$ (per thousand)

The fact that we have obtained a good adjustment with the life table for Mexico in 1940 must not delude us, however. We have contented ourselves with an approximate coincidence, and there are a number of other Malthusian populations which coincide quite as well as that described above.

By way of example, we have plotted in graph II.15 the cluster of points (similar to those in graph II.14) for two other life tables in the United Nations series: one for level 20 (expectation of life at birth for both sexes of 30 years) and one for level 100 (expectation of life at birth for both sexes of 70.2 years). In both cases, we have adjustment straight lines quite as good as those in graph II.14. Indeed, we should have good adjustment straight lines for all the intermediate model life tables between the two levels considered in graph II.15.



Graph II.15. Compatibility of the age distribution of the female population of Mexico, according to the 1940 census— with the two intermediate model life tables corresponding to expectations of life at birth for both sexes of 30 years (curve A) and 70.2 years (curve B)

Finally, for the three Malthusian populations considered in graphs II.14 and II.15 we have the following crude rates:

Crude rate (per thousand)	Life table for Mexico, 1940	*Intermediate model life table*	
		Level 20 (a)	*Level 100* (b)
b_0	51.29	45.71	45.71
d_0	26.42	32.59	8.41
r_0	24.87	13.12	37.30

(a) Expectation of life at birth for both sexes of 30 years.
(b) Expectation of life at birth for both sexes of 70.2 yea

The variations in the crude birth rate are not very great, and it can no doubt be accepted that the crude female birth rate in Mexico in 1940 was between 45 and 50 per thousand. However, there is considerable uncertainty regarding the crude death rate and rate of increase. Here we encounter for the first time a difficulty resulting from the fact that variations in mortality of the kind which occur in the human species [26] have little effect on the age structure of populations. Conversely,

---

[25] It may be recalled that Log $e = 0.43429$.

[26] It would perhaps be better to say "of the kind which have occurred in the human species", since there is no absolute certainty that mortality in countries where it is still high will decline in the same way as it did in the past in the developed countries.

therefore, very diverse assumptions regarding the unknown level of mortality can be compatible with the age structures observed.

The crude birth rate determined with the aid of graph II.15 (51.29 per thousand) seems a little high, because the crude birth rate recorded in 1940 was 44.3 per thousand, and if the true rate was 51.29 per thousand it would be necessary to assume under-registration of births of the order of 15 per cent. Although such a percentage is not unusual in developing countries, it is generally agreed that it would be unusual in the case of Mexico, where registration of births is considered to be almost total.

COMPATIBILITY OF A GIVEN DISTRIBUTION OF DEATHS WITH A GIVEN DEATH RATE

We can pose a problem similar to that dealt with above, using the age distribution of deaths instead of the age distribution of the population.

We can ask to what extent the observed age distribution of deaths $d(a)$ in Mexico in 1940 belongs to the sub-set $H_0(r)$ associated with the life table computed for Mexico for 1940.

If there were a perfect coincidence between the observed age distribution of deaths $d(a)$ and the age distribution of deaths of a particular Malthusian population of the sub-set $H_0(r)$, we should have:

$$d(a) = -\frac{b}{d}p'(a)e^{-ra}$$

and if we take the logarithms:

$$\text{Log}\,\frac{d(a)}{-p'(a)} = -ra\,\text{Log}\,e - \text{Log}\,\frac{d}{b}$$

It may be noted in passing that $-p'(a)$ is simply the age distribution of deaths in the stationary population corresponding to the life table for 1940.

If we assume that:

$$y = \text{Log}\,\frac{d(a)}{-p'(a)} \quad \text{and} \quad x = a$$

then the points (x, y) will be on the straight line:

$$y = -rx\,\text{Log}\,e - \text{Log}\,\frac{d}{b} \qquad \text{(II.26)}$$

Table II.18 gives details of the computation of y for the female population and graph II.16 shows that if we are satisfied with an approximate coincidence the pattern of points (x, y) can be satisfactorily adjusted by a straight line. As in graph II.14, the points which deviate from the straight line are those corresponding to young children (0 and 1-4 years of age) and old people (80 and over).

The ordinate at the origin of the straight line gives:

$$-\text{Log}\,\frac{d}{b} = 0.330$$

whence

$$\text{Log}\,\frac{d}{b} = 1.670$$

and finally

$$\frac{d}{b} = 0.46774$$

or

$$\frac{b-r}{b} = 1 - \frac{r}{b} = 0.46774$$

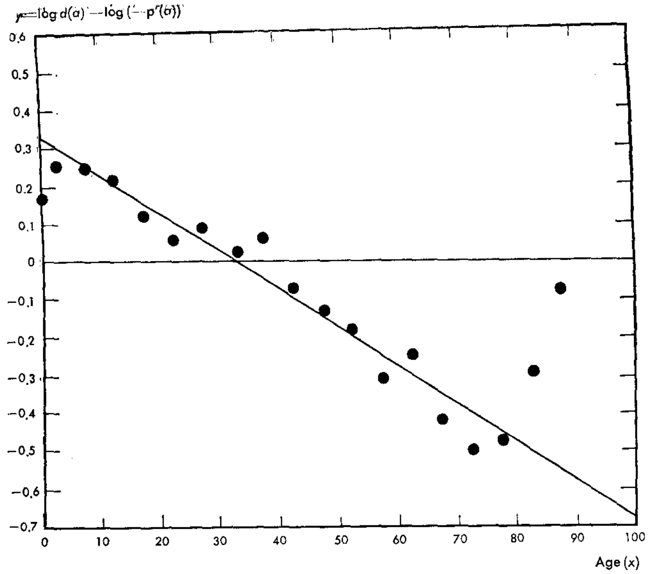
whence

$$\frac{r}{b} = 0.53226$$

TABLE II.18. CALCULATION OF THE QUANTITIES y OF FORMULA (II.25) IN THE TEXT WITH A VIEW TO STUDYING THE COMPATIBILITY OF THE AGE DISTRIBUTION OF FEMALE DEATHS RECORDED IN MEXICO IN 1940 ($d(a)$) WITH THE FEMALE LIFE TABLE COMPUTED FOR MEXICO FOR THE SAME YEAR ($p(a)$)

Age group (years)	*Distribution of female deaths by age groups observed in 1940* $d(a)$	*Distribution of deaths in the stationary population* $-p'(a)$	Log $d(a)$	Log $[-p'(a)]$	*Difference* Log $d(a)$ — Log $[-p'(a)]$
Under 1 year .	22 604	15 072	4.35419	4.17644	0.17775
1-4	25 297	13 794	4.40307	4.13969	0.26338
5-9	4 922	2 751	3.69214	3.43949	0.25265
10-14	2 020	1 216	3.30535	3.08493	0.22042
15-19	2 699	2 001	3.43120	3.30125	0.12995
20-24	3 184	2 792	3.50297	3.44592	0.05705
25-29	3 506	2 828	3.54481	3.45148	0.09333
30-34	3 184	2 998	3.50297	3.47683	0.02614
35-39	3 699	3 221	3.56808	3.50799	0.06009
40-44	2 966	3 490	3.47217	3.54283	—0.07006
45-49	2 789	3 762	3.44545	3.57542	−0.12997
50-54	2 714	4 191	3.43361	3.62232	−0.18871
55-59	2 367	4 738	3.37420	3.67560	−0.30140
60-64	3 781	6 633	3.57761	3.82171	−0.24410
65-69	2 740	7 128	3.43775	3.85297	−0.41522
70-74	3 214	8 181	3.50705	3.91281	−0.49576
75-79	2 210	6 498	3.34439	3.81278	−0.46839
80-84	2 551	5 051	3.40671	3.70338	−0.29667
85 + ...	3 553	3 615	3.55059	3.55811	−0.00752
	100 000	100 000			

SOURCES: For the age distribution of deaths, see *Demographic Yearbook, 1951* (United Nations publication, Sales No.: 52.XIII.1). For the life table, see *Demographic Yearbook, 1961* (United Nations publication, Sales No.: 62.XIII.1).



Graph II.16. Compatibility of the age distribution of female deaths observed in Mexico in 1940, $d(a)$, with the female life table computed for Mexico in 1940, $p(a)$

The abscissa of the point corresponding to 100 on the adjustment line is:

$-0.680 = -100r \text{ Log } e + 0.330$, whence $r = 0.02326$.

When we know r we can calculate b and d by the formulae given earlier, so that we finally have:

$b = 0.04370$
$d = 0.02044$
$r = 0.02326$

Compared with the result of graph II.14, we have approximately the same value for the rate of increase, r, but the crude birth and death rates are different, the crude birth rate being this time a little too low.[27]

Effects of census and registration errors

We have used crude data just as they were collected, but we know that these data suffer from certain errors, including, in particular:

(*a*) Under-estimation of the number of very young persons (particularly children under 1 year of age and, to a less extent, children between the ages of 1 and 4);

(*b*) Exaggeration of the ages of elderly persons (80 or over). These errors affect the data both for the number of persons in the population and for the number of deaths, so that between the ages of 5 and 80 the values $C(a)$ and $d(a)$ used to construct graphs II.14 and II.16 are lower than they should be.

We have also used for these graphs the life table calculated for 1940 (function $p(a)$ for graph II.14 and function $-p'(a)$ for graph II.16). This table has, however, been at least partially corrected for the errors mentioned.[28] Correction of the data would thus have the result, between the ages of 5 and 80, of raising the ordinates of the points in graph II.14 and lowering the ordinates of the points in graph II.16, and the two characteristics of the Malthusian populations determined from the age structure and the age distribution of deaths respectively would consequently be closer to each other. In other words, the two values obtained for the crude birth rates would probably straddle the true rates. The mean figure:

$$\frac{51.3 + 43.4}{2} = 47.5 \text{ (per thousand)}$$

is in fact in conformity with everything we know about the birth rate in Mexico in 1940.

What can be assumed regarding female fertility when there are not data for it?

In all the preceding examples, except that leading to the concept of a stable population, the fertility of the Malthusian populations has remained unknown throughout.[29] We know, however, that it must satisfy the equation:

$$\int_u^v e^{-ra} p_f(a) \varphi_f(a, t)\, da = 1 \qquad \text{(II.13)}$$

In discontinuous notation, this equation is written, for 100,000 girls at birth:

$$\sum_u^v e^{-ra} \mathrm{L}_a f_a(t) = 100\,000 \qquad \text{(II.13}b\text{)}$$

The fertility will be determined only if we assume an additional condition.

The variation in the female fertility rate as a function of the age of the woman always follows the same pattern, beginning from zero at about 15 years of age, rising to a maximum between 20 and 30, and then declining until it returns to zero at about the age of 50. The rising phase is dependent primarily on marital status, since at the ages below 50 few women are sterile and under conditions of natural fertility, i.e., when there is no birth control, a woman's reproductive activity is determined almost entirely by whether or not she is married. Even where there is birth control, it has little influence at the time when a family is being started, and marital status therefore remains the preponderant factor. The earlier marriage takes place, the earlier the age at which maximum fertility is reached.

The declining phase of the age-specific fertility rate depends mainly, under natural conditions, on the fact that more and more women become sterile as they grow older.[30] Where birth control is practised, the situation is affected by voluntary sterility through the use of contraceptive methods in addition to the natural phenomenon of increasing female sterility with increasing age.

Table II.19 gives the age distribution of fertility rates observed recently in various countries of the world. We see that this distribution varies relatively little, and the distributions observed in Jamaica in 1951 and in Spain in 1940 may be considered extremes. It might be thought surprising that the mean distributions in countries with high fertility are very close to those of countries with

---

[27] It may be recalled that the recorded crude rate was 44.3 per thousand and that, as we stated, there was a relatively small, but nevertheless not negligible, under-registration of births.

[28] For example, according to the recorded deaths the female infant mortality rate in 1940 was 118.1 per 1 000 live births, while in the life table it was 150.1 per 1 000 live births.

[29] In the example for a stable population, the data included the fertility.

[30] This does not necessarily mean biological fertility, for social factors also enter into the question. The adjective "natural" simply signifies that we are not concerned with sterility consciously connected with family size, as in the case of voluntary sterility through the use of contraceptive methods.

TABLE II.19. DISTRIBUTION OF AGE-SPECIFIC FERTILITY RATES (a) IN SELECTED AREAS

Age group (years)	*Average (b) for 52 countries*	*Average for 15 countries with high fertility*	*Average for 37 countries with low fertility*	*Jamaica (1951)*	*Spain (1940)*	*Intermediate distribution*
15-19 (c) . . .	63	93	51	136	14	100
20-29	253	251	254	292	147	273
25-29	276	235	285	249	303	263
30-34	211	196	216	166	272	188
35-39	134	137	132	110	179	121
40-44 (c)	63	69	60	47	85	55
15-44 years . .	1 000	1 000	1 000	1 000	1 000	1 000

(a) The distribution covers both male and female fertility, but as the rate of masculinity of births may be considered constant the distribution indicated is valid for female fertility rates.

(b) This average is calculated on the basis of all the data given in the *Demographic Yearbook, 1954* (United Nations publication, Sales No.: 54.XIII.1), table 11, pp. 283-294.

(c) Births to women under the age of 15 have been included in the 15-19 group, while births to women over the age of 45 have been included in the 40-44 group.

low fertility. However, what we have said regarding the factors governing observed distributions enables us to understand this apparent contradiction. The rate at which "natural" sterility increases with the age of women is by no means the same everywhere. In populations of European origin, for instance, the increase is much slower than in Asian, African or Latin American countries. Thus, where there is no birth control, we observe that the decline in the fertility rate with age, once the maximum has been passed, is much slower in Europe than elsewhere. The use of birth control practices compensates for this slowness, however, and this is why the declining phase of the fertility rates is similar in both groups of countries. The rising phase depends on marital status, and when we consider the average figures [31] we obtain variations which are more or less identical.

As a first approximation, we can define a model fertility for the human species by taking an invariable age distribution of fertility rates. If this is done, a given fertility is then determined by a single parameter, namely the total of the fertility rates or (which amounts to the same thing) the gross reproduction rate.

For reasons which will be explained later, it is convenient to adopt, when defining this model fertility, the distribution given in the last column of table II.19, which has been termed the "intermediate distribution". In fact, distribution has already been used several times in the study of stable populations.

If F_a represents this intermediate distribution and $R'(t)$ represents the gross reproduction rate, we have:

$$R'(t)F_a = 5f_a(t)$$

and equation II.13 is written

$$R'(t)\sum_u^v e^{-ra}L_aF_a = 500\,000$$

whence

$$R'(t) = \frac{500\,000}{\sum_u^v e^{-ra}L_aF_a}$$

which is not dependent on time. Thus, the assumption that fertility follows a well-defined pattern of variation enables us to determine the model fertility function, once the other characteristics of the Malthusian population are known.

Here, by way of illustration, are details of how the fertility function is calculated in the first example of the determination of a Malthusian population, where the rate of increase was assumed to be known. Table II.20 gives details of the computations. Once we know the gross reproduction rate R′, we obviously obtain the age-specific fertility rates through the formula

$$f_a = \frac{R'\,F_a}{5}$$

TABLE II.20. CALCULATION OF THE FERTILITY OF EXAMPLE (a) No. 1 (INTERMEDIATE MODEL FERTILITY)

Median age a	*Age group (years)*	$e^{-ra}L_a$ *(figures taken from table II.3)*	*Distribution of fertility rates* F_a	*Product of the two preceding columns* $e^{-ra}L_aF_a$
17.5 . .	15-19	232 110	0.100	23 211
22.5 . .	20-24	194 686	0.273	53 169
27.5 . .	25-29	162 447	0.263	42 724
32.5 . .	30-34	135 620	0.188	25 497
37.5 . .	35-39	112 394	0.121	13 600
42.5 . .	40-44	93 083	0.055	5 120
All ages			1.000	163 321

(a) Malthusian population with a rate of increase $r = 0.03$ in the sub-set $H_0(r)$ associated with the intermediate model life table giving an expectation of life at birth for both sexes of 50 years.

From the table we know that:

$$R' = \frac{500\,000}{163\,321} = 3.068$$

However, the model fertility described above, which is based on the assumption of an invariable age distribution of fertility rates, is of course, simply one of many methods. It merely represents a first approximation based on the consideration of averages computed for a large number of countries. It is quite certain that, for a country with a given nuptiality and "natural" sterility, the ever-wider

---

[31] This does not apply, of course, if we consider individual countries.

TABLE II.21. MODEL FERTILITY TABLES (FERTILITY RATES CALCULATED PER 1,000 WOMEN ON THE BASIS OF BOYS AND GIRLS TOGETHER)

Age group (years)	*Observed fertility rate in Chile* (a) *in 1952*	*Gross reproduction rate*				
		2.50	*2.75*	*3.00*	*3.25*	*3.50*
		I. Late fertility with slow decline (rate per 1 000)				
15-19 . . .	75.1	78.1	82.6	86.9	90.8	94.6
20-24 . . .	223.7	235.4	252.8	269.8	286.2	302.0
25-29 . . .	232.2	247.0	269.6	292.1	314.2	335.9
30-34 . . .	192.3	206.8	229.6	252.3	275.2	298.1
35-39 . . .	145.3	158.0	178.2	198.8	219.9	241.3
40-44 . . .	71.1	78.3	89.7	101.5	113.8	126.5
45-49 . . .	19.2	21.4	24.9	28.6	32.5	36.6
	Puerto Rico (b) *in 1950*	*II. Early fertility with rapid decline (rate per 1 000)*				
15-19 . . .	99.2	96.3	103.8	109.4	114.7	120.0
20-24 . . .	279.7	272.6	295.5	317.4	337.8	357.3
25-29 . . .	260.3	254.7	279.5	303.7	327.8	351.8
30-34 . . .	200.0	196.3	217.5	240.0	262.6	285.5
35-39 . . .	143.1	141.0	158.7	176.8	196.2	216.0
40-44 . . .	53.1	52.5	59.3	67.4	75.9	84.7
45-49 . . .	11.7	11.6	13.2	15.3	17.5	19.7

(a) Gross reproduction rate observed in 1952 in Chile: R′ = 2.339.
(b) Gross reproduction rate observed in 1950 in Puerto Rico: R′ = 2.554.

use of contraceptive practices cannot be reconciled with an invariable age distribution of fertility rates.

Observation shows that, in such circumstances, it is the rates of fertility at the higher ages which begin to decline, and this diminution also increasingly affects the rates at earlier ages.

The Regional Centre for Demographic Training and Research in Latin America at Santiago, Chile, which operates under United Nations auspices within the framework of the technical assistance programme, attempted on this basis to define a number of fertility models where the age distribution of fertility rates varied with the level of fertility.

A series of fertility rates for relatively late fertility with a slow decline at the higher ages was determined on the basis of observed fertility in Chile in 1952, while a series of fertility rates for relatively early fertility with a rapid decline at the higher ages was prepared on the basis of observed fertility in Puerto Rico in 1950. These rates are given in table II.21. The rates in question are calculated for girls and boys together. In order to obtain the female rates, the rates given must be divided by 2.05. The distributions at extreme ages are given in table II.22, together with some other distributions already encountered.

Let us take up the preceding problem, using the model fertility determined on the basis of the Chilean fertility. We have the equation:

$$\sum_u^v e^{-ra} L_a f_a(t) = 100\ 000.$$

Let us calculate the results of this equation for various values of R′ and find out by interpolation the value of R′ for which the integral equals 100,000. Details of the calculation are given in table II.23. We find that R′ equals 3.28. From R′ we calculate by interpolation the

TABLE II.22. AGE DISTRIBUTION OF FERTILITY RATES FOR MODEL TABLES OF EXTREME FERTILITY FROM TABLE II.21 AND COMPARISON OF THOSE RATES WITH THE INTERMEDIATE DISTRIBUTION, THE DISTRIBUTIONS OBSERVED IN JAMAICA IN 1951 AND SPAIN IN 1940, AND THE AVERAGE DISTRIBUTION FOR 52 COUNTRIES

Age group (years)	*Gross reproduction rate*		*Intermediate distribution* (a)	*Spain 1940* (a)
	2.50	*3.50*		
	Late fertility with slow decline			
15-19 . . .	76	66	100	14
20-24 . . .	230	210	273	147
25-29 . . .	241	234	263	303
30-34 . . .	202	208	188	272
35-39 . . .	154	168	121	179
40-44 . . .	76	88	55	85
45-49 . . .	21	26		
ALL AGES .	1 000	1 000		

Age group (years)	*Gross reproduction rate*		*Average for 52 countries* (a)	*Jamaica 1951* (a)
	2.50	*3.50*		
	Early fertility with rapid decline			
15-19 . . .	94	84	63	136
20-24 . . .	266	249	253	292
25-29 . . .	249	245	276	249
30-34 . . .	191	199	211	166
35-39 . . .	138	150	134	110
40-44 . . .	51	59	63	47
45-49 . . .	11	14		
ALL AGES .	1 000	1 000	1 000	1 000

(a) Figures taken from table II.19.

TABLE II.23. CALCULATION OF THE FERTILITY OF THE POPULATION OF EXAMPLE No. 1. LATE FERTILITY WITH SLOW DECREASE, BASED ON OBSERVED FERTILITY IN CHILE IN 1952 (SEE TABLE II.21)

Median age *a* *(1)*	*Age group (years)* *(2)*	$e^{-ra}L_a$ *(3)*	*Product of column 3 by the female fertility rates corresponding, in table II.21, to the following gross reproduction rates:* (a)		
			3.00	*3.25*	*3.50*
17.5	15-19	232 110	9 839	10 280	10 170
22.5	20-24	194 686	25 623	27 180	28 680
27.5	25-29	162 447	23 150	24 900	26 620
32.5	30-34	135 260	16 691	18 206	19 720
37.5	35-39	112 394	10 900	12 056	13 229
42.5	40-44	93 083	4 608	5 167	5 743
47.5	45-49	76 500	1 068	1 251	1 367
ALL AGE .			91 879	99 040	106 069

By interpolation. we see that the amount in the last line is equal to 100 000 for R′ = 3.28.

(a) In table II.21, the fertility rate in question is the one computed on the basis of all male and female births together. In order to obtain the female fertility rates, the rates given in table II.21 should be divided by 2.05 (this assumes a masculinity at birth of 105).

values of the corresponding fertility rates in table II.21. We find the following rates (per 1,000 women):

Age group	*Fertility rate*
15-19	91.3
20-24	288.4
25-29	317.2
30-34	279.1
35-39	222.8
40-45	115.5
45-49	33.1

Here again, the rates are not time-dependent. The fertility can be defined in accordance with the assumption that it is of the same form as the model fertility defined at the top of table II.21.

We should not, however, lose sight of the fact that the result obtained depends on the model fertility used in the calculation. Table II.24 gives some indication of the divergences which may be encountered in the determination of the gross reproduction rates (R′) if we take one model fertility rather than another. In these comparative calculations we have taken the sub-set H₀(*r*), the intermediate model life table giving an expectation of life at birth for both sexes of 60.4 years, and a rate of natural increase of *r* = 3.5 per cent. It will be noted that what we called the "special case" gives a gross reproduction rate R′ which lies between the extreme values corresponding to the distributions observed in Jamaica and Spain. We can obviously determine the distributions which give exactly the same result as the "special case". The values will coincide only for a single value of the gross reproduction rate, but threre are grounds for hoping that the divergences will not be very great for other levels of fertility. It was in this way that we determined what we termed the "intermediate distribution". This method gives *approximately* the same result as the "special case". Thus, in table II.24, the "special case" gives R′ = 3.066, and the intermediate distribution R′ = 3.042. Other examples of approximate coincidences for other fertility

TABLE II.24. SIX SERIES OF FEMALE FERTILITY RATES (a) (PER 1 000 WOMEN) LEADING TO A STABLE POPULATION WITH AN INTRINSIC RATE OF NATURAL VARIATION OF 3.5 PER CENT WHEN ASSOCIATED WITH AN INTERMEDIATE MODEL LIFE TABLE CORRESPONDING TO AN EXPECTATION OF LIFE AT BIRTH FOR BOTH SEXES OF 60.4 YEARS

Age group (years)	*Nature of the model fertility table (births of both sexes)*					
	Constant distribution (Jamaica, 1951)	*Intermediate constant distribution*	*Special case*	*Relatively early fertility*	*Relatively late fertility*	*Constant distribution (Spain, 1940)*
15-19	164.5	124.7		112.0	91.2	19.6
20-24	353.2	340.5		327.4	287.8	206.5
25-29	301.1	328.0	1 256,9	315.6	316.4	425.6
30-34	200.8	234.5		251.1	277.5	382.1
35-39	133.0	150.9		185.3	221.9	251.4
40-44	56.8	68.6		71.6	115.1	119.4
45-49				16.4	32.9	
Gross reproduction rate . . .	2.950	3.042	3.066	3.120	3.275	3.424
Net reproduction rate	2.518	2.590	2.619	2.651	2.767	2.888
Intrinsic rate of natural variation (percentage)	3.5	3.5	3.5	3.5	3.5	3.5
Expectation of life at birth for both sexes (years) . . .	60.4	60.4	60.4	60.4	60.4	60.4

(a) These rates are computed on the basis of all births (girls and boys).

levels will be given in chapter VII. This intermediate distribution was determined by trial and error, and it depends to some extent on how the method of trial and error has been carried out. Thus, other distributions could give results which are at least as good, and perhaps better.

RECONSIDERATION OF THE SPECIAL CASE OF STABLE POPULATIONS

As was pointed out above, the calculation of the characteristics of a stable population was greatly simplified when the fertility function $\varphi(a)$ was reduced to a single value $\varphi(27.5)$, which was then equal to the gross reproduction rate R'. We stressed the artificial nature of such a fertility function.

Let us disregard this artificiality for a moment, however, and consider only the result, i.e., the population calculated by these means. This is a population in which:

(*a*) The age distribution is constant;
(*b*) The mortality is constant and known;
(*c*) The rate of increase is known.

Thus, it is the Malthusian population of the sub-set $H_0(r)$ associated with a known mortality function whose rate of increase r_0 is also known. We have seen that the fertility of such a population is determined if we assume knowledge of the age distribution of fertility rates, which we assume to be constant. This fertility can be that of the special case used in the calculation, but it can also be very different, and its level then depends on the age distribution of fertility rates adopted, although if we adopt the intermediate distribution, the level of fertility is very close to that corresponding to the special case.

In other words, if we assume the mortality function and the value of the gross reproduction rate R', and if we calculate two stable populations by applying the simplified method of the special case on the one hand and the general method, using the intermediate age distribution of fertility, on the other, the two populations thus arrived at are very close to each other. The method of the special case is then revealed as a device for the easy calculation of stable populations having an intermediate age distribution of fertility rates.